UNIVERSITY OF BATH

Citation for published version:
Boden, A & Matthies, K 2014, 'Existence and homogenisation of travelling waves bifurcating from resonances of reaction-diffusion equations in periodic media', Journal of Dynamics and Differential Equations, vol. 26, no. 3, pp. 405-459. https://doi.org/10.1007/s10884-014-9398-6

DOI:
10.1007/s10884-014-9398-6

Publication date:
2014

Document Version
Peer reviewed version

Link to publication

EXISTENCE AND HOMOGENISATION OF TRAVELLING WAVES BIFURCATING FROM RESONANCES OF REACTION-DIFFUSION EQUATIONS IN PERIODIC MEDIA

ADAM BODEN AND KARSTEN MATTHIES

Abstract. The existence of travelling wave type solutions is studied for a scalar reaction diffusion equation in $\mathbb{R}^2$ with a nonlinearity which depends periodically on the spatial variable. We treat the coefficient of the linear term as a parameter and we formulate the problem as an infinite spatial dynamical system. Using a centre manifold reduction we obtain a finite dimensional dynamical system on the centre manifold with fully degenerate linear part. By phase space analysis and Conley index methods we find conditions on the parameter and nonlinearity for the existence of travelling wave type solutions with particular wave speeds.

The analysis provides an approach to the homogenisation problem as the period of the periodic dependence in the nonlinearity tends to zero.

1. Introduction

In this paper we investigate the existence of travelling wave type solutions for a reaction diffusion equation in $\mathbb{R}^2$ with a nonlinearity which is periodically dependent on the spatial variable. Specifically, we will consider the equation

$$u_t = \operatorname{div}(A\nabla u) + f\left(\frac{x}{\varepsilon}, u\right), \tag{1}$$

where A is a real symmetric positive definite matrix, $\varepsilon > 0$ and the nonlinearity $f(\xi, u)$ is periodic in ξ with periodic cell $[0, 2\pi]^2$, i.e.

$$f(\xi_1 + \xi_2, u) = f(\xi_1, u) \text{ for all } \xi_2 \in (2\pi\mathbb{Z})^2 .$$

In particular we consider a class of nonlinearities of the form

$$f(\xi, u) = -\mu u + p(\xi)q(u), \tag{2}$$

where $\mu \in \mathbb{R}$, $p \in H^2\left(T^2\right)$ the periodic Sobolev space on $[0, 2\pi]^2$ and $q \in C^2(\mathbb{R})$ with $q(0) = 0$, $q'(0) = 0$ and $q''(0) \neq 0$.

For the reaction diffusion equation with a nonlinearity in this class we are interested in the existence of travelling wave type solutions: A solution is a classical travelling wave solution if it has the form

$$u(x, t) = v\left(x \cdot k - ct\right), \tag{3}$$

where $k \in S^1$, $c \neq 0$ is a constant and $v = v(\tau)$ is a fixed profile. Hence they are solutions which are fixed orthogonal to k and move in the k direction with speed c as time varies. However due to the periodic behaviour in the nonlinearity such travelling wave solutions will not in general exist for equation (1). Therefore we need to modify the type of solution we look for to accommodate this periodic behaviour. Thus we look for generalised travelling wave solutions of the form

$$u(x, t) = v^\varepsilon\left(x \cdot k - ct, \frac{x}{\varepsilon}\right), \tag{4}$$

where the profile function $v^\varepsilon = v^\varepsilon(\tau, \xi)$ is periodic in ξ with periodic cell $[0, 2\pi]^2$. This type of travelling wave solution has a profile which varies as it moves over the periodic cells and therefore is able to incorporate the effects of the periodic dependence in the nonlinearity into the solution.

The first result we will show is that bounded generalised travelling wave solutions exist for the reaction diffusion equation with a nonlinearity of form (2); provided the parameter μ is sufficiently close to certain eigenvalues of the diffusion term and the function p satisfies certain conditions. Thus we find generalised travelling wave solution when the parameter μ is close to cancelling certain eigenvalues of the diffusion terms.

Once we have established the existence of these generalised travelling wave solutions in a variety of different cases, we will then look at a particular case where we have proved existence and examine what happens to these generalised travelling wave solutions as $\varepsilon \to 0$. For this particular case we will show that there exist a limiting profile $v^0(\tau)$ such that

$$v^\varepsilon\left(x \cdot k - ct, \frac{x}{\varepsilon}\right) = v^0\left(x \cdot k - ct\right) + O(\varepsilon) \text{ as } \varepsilon \to 0.$$

This result tells us that the generalised travelling wave solution are of order ε close to a fixed profile.

The idea of travelling wave solutions was first introduced in the work of Kolmogorov, Petrovsky and Piskunov [15] and Fisher [10] for a homogeneous reaction diffusion equation.

Subsequently travelling wave solution for reaction diffusion equations and reaction diffusion advection equation in periodic media were introduced and investigated. A detailed discussion of this work can be found in the paper [25] and book [26] by Xin or the paper [3] by Berestycki and Hamel. The methods that Xin and, Berestycki and Hamel, use to prove the existence and monotonicity of travelling wave solutions in periodic media are based on the use of the maximum principle and the method of moving planes (which is described in the paper by Berestycki and Nirenberg [4]). These methods allow the existence of travelling wave solutions to be proved in a wide variety of cases (details can be found in [3] and [26]). However because of the use of the maximum principle these methods have problems with systems of reaction diffusion equations. The method we will use in this paper does not allow use to obtain as general detailed as those of Berestycki and Hamel, but it will naturally generalise to systems of reaction diffusion equations. The approach we use is to formulate the problem as a spatial dynamical system and use a centre manifold reduction. Thus we start by substituting the generalised travelling wave ansatz (4) into the reaction diffusion equation to obtain a degenerate second order elliptic equation on a infinite cylinder with periodic boundary conditions in terms of the profile function v. After this we use the idea of Kirchgässner [14] to formulate this equation as a infinite dimensional dynamical system treating the unbounded direction as a time variable. This idea has been used in many papers to find and analyse solutions to partial differential equations on unbounded domains a few examples are [1, 8, 9, 12, 20, 21] and most relevantly [16], and [17], which used it to study travelling wave solutions and pinning for a reaction diffusion equation on a infinite cylinder with periodic behaviour along the cylinder.

We then apply the local centre manifold theorem to this spatial dynamical system, this requires some work as the linear part of our spatial dynamical system is not bisectorial. This theorem allows us to find solutions for our dynamical system by investigating the dynamics on a finite dimensional local manifold about the origin. Details about this theorem can be found in the paper by Vanderbauwhede and Iooss [24] and the book by Haragus and Iooss [11]. After applying the local centre manifold reduction we break the problem into two cases:

Firstly if the parameter μ in the nonlinearity is close to zero then the problem reduces to a one dimensional centre manifold. Then we can study the dynamics on the centre manifold directly to find conditions for the existence of bounded generalised travelling wave solutions.

The other case we consider is when the parameter μ is close to a non-zero eigenvalue which leads to a two dimension centre manifold. in this case we use Conley index to obtain conditions on the nonlinearity for the existence of bounded generalised travelling wave solutions. Details about Conley index can be found in [7, 22].

Finally to study the homogenisation problem we formulate the problem as a dynamical system in a slightly different way, so that we can control how the reduction converges as $\varepsilon \to 0$. Using this information we then study the convergence of the dynamics on the centre manifold as $\varepsilon \to 0$.

The remainder of this paper is organised into a series of section and subsection in which we present and use the techniques outlined above to prove our results.

In the next section we state the main theorems : Firstly we state a result dealing with the existence of generalised travelling wave solutions. Then we state a result which deals with

the homogenisation of the travelling wave solution for a particular case where we have proved existence. The proof of the existence result is spread over the next two sections: In section 3 we prove the existence of a centre manifold reduction for our infinite dimensional dynamical system. Then in section 4 we analyse the ordinary differential equation on the centre manifold to prove the existence of generalised travelling wave solutions. The homogenisation result is proved in the final section 5.
The results in this paper are taken from the Phd thesis of Boden [5] which was done under the supervision of Matthies.

2. Results

For the nonlinearities as in (2) we treat μ as a parameter and investigate the values of μ, which create the bifurcations of generalised travelling wave solutions. Such solutions exist if the μ is close to certain eigenvalues of the diffusion term.

Theorem 2.1. *Let $\varepsilon > 0$, $c \neq 0$, and $k \in S^1$ be fixed and*

$$\mu = -\frac{1}{\varepsilon^2}\eta^T A\eta - \delta \tag{5}$$

for $\eta \in \mathbb{Z}^2$ and $\delta \in \mathbb{R}$ with $|\delta| > 0$ sufficiently small, then we have the following results:

(1) If $\eta = 0$ and $p \in H^2\left(T^2\right)$ is such that $\int_{T^2} p(s)ds \neq 0$, then there exists a generalised travelling wave solution corresponding to a heteroclinic connection between equilibria i.e.

$$v(\tau,\xi) \to v^{\pm}(\xi) \text{ as } \tau \to \pm\infty.$$

(2) If $\chi(\eta) := \#\left\{m \in \mathbb{Z}^2 : m^T Am = \eta^T A\eta\right\} = 2$ and p is in a non-empty open subset of $H^2\left(T^2\right)$, then there exists a non-trivial generalised travelling wave solution. Furthermore with additional restrictions on p there is a non-empty open subset of $H^2\left(T^2\right)$, where this solution will be a heteroclinic connection between equilibria.

Remark 2.2.

- The values $-\eta^T A\eta/\varepsilon$ are eigenvalues of the diffusion term $\operatorname{div}(A\nabla\cdot)$ with eigenvectors $\exp\left(\frac{\mathrm{i}\eta\cdot x}{\varepsilon}\right)$
- The equilibria $v^{\pm}(\xi)$ are stationary solution of the reaction diffusion equation, thus $u(x) = v^{\pm}(x/\varepsilon)$ solves
$$0 = \operatorname{div}(A\nabla u) - \mu u + p\left(\frac{x}{\varepsilon}\right)q(u)$$
- The value of $\chi(\eta)$ determines the dimension of the centre manifold, in case (1) when $\eta = 0$ the dimension is $\chi(0) = 1$.
- By non-trivial solution we mean a solution that varies in τ.

Now that we have established the existence of generalised travelling wave solution, we turn our attention to the limit as the period $\varepsilon \to 0$. Our aim will be to show that the generalised travelling wave solutions which depend periodically on x can be approximated by a homogeneous travelling wave solution which does not depend periodically on x. This is related to results of a homogenisation type. For general theory of homogenisation see e.g. [2, 6, 13]. In the following theorem we prove such a result for the case when $\chi(\eta) = 1$, i.e. $\mu = -\delta$ is a small non-zero parameter. For this case we have the existence of a travelling wave solution which corresponds to a heteroclinic connection between equilibria.

Theorem 2.3. *Let $c \neq 0$ and $k \in S^1$ be fixed, $p_0 := \frac{1}{|T^2|}\int_{T^2} p(s)ds \neq 0$ and $\mu = -\delta \in \mathbb{R}$ be fixed with $|\delta| > 0$ sufficiently small. Then the generalised travelling wave profiles satisfy*

$$v^{\varepsilon}(\tau,\xi) = v^0(\tau) + O(\varepsilon) \text{ as } \varepsilon \to 0$$

uniformly on $\mathbb{R}$, where the limiting profile v^0 is a heteroclinic connection between equilibria which satisfies the ordinary differential equation

$$v^0_\tau = -\frac{\delta v^0 + p_0 q(v^0)}{c}.$$

3. Reduction to Finite Dimensions

The approach we take, to prove theorem 2.1, is to first find an equation for the profile function v and write it as a spatial dynamical system, using the idea of Kirchgässner [14]. We will then use a local centre manifold reduction to reduce our problem to finding solutions on a finite dimensional manifold. In this section we show that such a centre manifold reduction exist for our problem.

3.1. Spatial Dynamical System Formulation.

We obtain a spatial dynamical system formulation by substituting the ansatz

$$u(x,t) = v\left(x \cdot k - ct, \frac{x}{\varepsilon}\right) \tag{4}$$

into the reaction diffusion equation (1). This leads to an equation in terms of the profile function v

$$-cv_\tau = \frac{1}{\varepsilon^2}\mathrm{div}_\xi\left(A\nabla_\xi v\right) + \frac{2}{\varepsilon}k^T A\nabla_\xi v_\tau + v_{\tau\tau} - \mu v + p(\xi)q(v), \tag{6}$$

which is a second order elliptic partial differential equation for $(\tau,\xi) \in \mathbb{R} \times [0,2\pi]^2$ with periodic boundary conditions.

We formulate this equation as a spatial dynamical system by treating the unbounded direction τ as time and the $\delta \in \mathbb{R}$ which appears in μ in (5) as an extra dependent variable. Thus if we let $U = (v, v_\tau)$, we obtain the spatial dynamical system

$$\begin{aligned} U_\tau &= \mathcal{A}U + F(U,\delta) \\ \delta_\tau &= 0, \end{aligned} \tag{7}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi\left(A\nabla_\xi \cdot\right) + \eta^T A\eta\right) & -c - \frac{2}{\varepsilon}k^T A\nabla_\xi \end{bmatrix} \in \mathcal{L}(X,Z)$$

and

$$F(U,\delta) = F((u_1,u_2),\delta) = \begin{bmatrix} 0 \\ -\delta u_1 - p(\cdot)q(u_1) \end{bmatrix} \in C^2(X\times\mathbb{R}, X);$$

where X and Z are Hilbert spaces of periodic functions, which will be defined in the subsection 3.3.

3.2. Centre Manifold Reduction.

Now we will show that there exists a local centre manifold reduction for the spatial dynamical system introduced in the previous section.

For the spatial dynamical system (7) the existence of a local centre manifold reduction means that there exists a local manifold around $(0,0) \in X \times \mathbb{R}$ on which we find solutions to our dynamical system. The idea when constructing this manifold is to first split the phase space $X \times \mathbb{R}$ into two parts $X_c \times \mathbb{R}$, where the spectrum of $\mathcal{A}$ has zero real parts, and X_h where the spectrum of $\mathcal{A}$ has non-zero real part. Once this is done we construct a map $\psi : X_c \times \mathbb{R} \to X_h$ and an open neighbourhood Ω of $(0,0) \in X \times \mathbb{R}$ such that the local manifold is $\mathcal{M}_c := \{(U^c + \psi(U^c,\delta), \delta) : (U^c,\delta) \in X_c \times \mathbb{R}\} \cap \Omega$.

Proposition 3.1. *There exist a finite dimensional subspace $X_c \times \mathbb{R} \subset X \times \mathbb{R}$ and a projection π_c onto X_c. Letting $X_h = (Id - \pi_c)X$, there exists a neighbourhood of the origin $\Omega \subset X \times \mathbb{R}$ and a map $\psi \in C_b^2(X_c \times \mathbb{R}, X_h)$ with $\psi(0,0) = 0$ and $D\psi(0,0) = 0$, such that if $(U^c,\delta) : I \to X_c \times \mathbb{R}$ solves*

$$\begin{aligned} U^c_\tau &= \mathcal{A}U^c + \pi_c F(U^c + \psi(U^c,\delta),\delta), \\ \delta_\tau &= 0 \end{aligned}$$

for some interval $I \subset \mathbb{R}$, and $(U,\delta)(\tau) = (U^c(\tau) + \psi(U^c(\tau),\delta(\tau)),\delta(\tau)) \in \Omega$ for all $\tau \in I$ then (U,δ) solves

$$\begin{aligned} U_\tau &= \mathcal{A}U + F(U,\delta) \\ \delta_\tau &= 0. \end{aligned}$$

Remark 3.2. Solutions found using this proposition lie on the local centre manifold

$$\mathcal{M}_c := \{(U^c + \psi(U^c, \delta), \delta) : (U^c, \delta) \in X_c \times \mathbb{R}\} \cap \Omega.$$

This result is just a reformulation of the local centre manifold theorem, which can be found in [24, Theorem 3] and [11, Theorem 2.9], for our particular case. Thus to prove this result we just need to check that the hypotheses of the local centre manifold theorem are satisfied, where we define space of exponentially growing Banach space E-valued continuous functions is defined as

$$C_\eta(\mathbb{R}, E) := \left\{ u \in C(\mathbb{R}, E) : \|u\|_\eta := \sup_{\tau \in \mathbb{R}} \left(\mathrm{e}^{-\eta|\tau|} \|u(\tau)\|_E \right) < \infty \right\}.$$

Hence we need to check the following hypotheses.

(H1) $\mathcal{A} \in \mathcal{L}(X, Z)$ and for some $k \geqslant 2$ there exists a neighbourhood of the origin $V \subset X$ such that $F(\cdot, \delta) \in C^k(V, X)$ and $F(0,0) = 0$, $D_U F(0,0) = 0$.

(H2) There exists a projection $\pi_c \in \mathcal{L}(Z, X)$ onto a finite dimensional subspace $Z_c = X_c \subset X$ such that $\sigma(\mathcal{A}|_{X_c}) \subset \mathrm{i}\mathbb{R}$ and

$$\mathcal{A}\pi_c U = \pi_c \mathcal{A} U,$$

for all $U \in X$.

(H3) Letting $X_h := (I - \pi_c)X$, there exists a $\gamma > 0$ such that for each $\eta \in [0, \gamma)$ and $f \in C_\eta(\mathbb{R}, X_h)$ the hyperbolic affine problem

$$\dot{U}_h = \mathcal{A}x_h + f \text{ and } U_h \in C_\eta(\mathbb{R}, X_h)$$

has a unique solution $x_h = K_h f$ where $K_h \in \mathcal{L}(C_\eta(\mathbb{R}, X_h))$ for each $\eta \in [0, \gamma)$ and

$$\|K_h\|_{\mathcal{L}(C_\eta(\mathbb{R}, X_h))} \leqslant \Gamma(\eta)$$

for some continuous function $\Gamma : [0, \gamma) \to \mathbb{R}^+$.

The assumption (H1) will be checked in subsection 3.4. In the subsections which follow we check each of the other hypotheses for δ fixed and then extend the result for the restricted system to the full system.

3.3. **Function Spaces.** We first need to define the phase spaces of our problem using eigenvalues and eigenfunctions of $\mathcal{A}$. Hence we consider the eigenvalue equation

$$\mathcal{A}U = \lambda U \text{ for } \lambda \in \mathbb{C}$$

and we look for eigenfunctions U of the form

$$U = \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \exp(\mathrm{i}m \cdot \xi) \text{ for } m \in \mathbb{Z}^2.$$

A function of this form is an eigenfunction if λ satisfies

$$\lambda^2 + \left(c + \frac{2\mathrm{i}}{\varepsilon} k^T A m \right) \lambda + \frac{1}{\varepsilon^2} \left(\eta^T A \eta - m^T A m \right) = 0. \tag{8}$$

Hence we find eigenvalues and eigenfunctions

$$\lambda_m^\pm := \frac{-\left(c + \frac{2\mathrm{i}}{\varepsilon} k^T A m\right) \pm \sqrt{\left(c + \frac{2\mathrm{i}}{\varepsilon} k^T A m\right)^2 + \frac{4}{\varepsilon^2}\left(m^T A m - \eta^T A \eta\right)}}{2}, \tag{9}$$

$$U_m^\pm := \begin{bmatrix} 1 \\ \lambda_m^\pm \end{bmatrix} \exp(\mathrm{i}m \cdot \xi) \text{ for } m \in \mathbb{Z}^2. \tag{10}$$

Now we can define the Hilbert space which will be the phase space for our analysis, for this definition and throughout this chapter we will use the summing conventions

$$\sum_{m \in \mathbb{Z}^2} a_m^\pm := \sum_{m \in \mathbb{Z}^2} a_m^+ + \sum_{m \in \mathbb{Z}^2} a_m^- \text{ and } \sum_{m \in \mathbb{Z}^2} \pm a_m^\pm := \sum_{m \in \mathbb{Z}^2} a_m^+ - \sum_{m \in \mathbb{Z}^2} a_m^-.$$

Thus we define the space

$$Z := \left\{ U = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm} : \alpha_m^{\pm} \in \mathbb{C}, \overline{\alpha_m^{\pm}} = \alpha_{-m}^{\pm} \text{ and } \sum_{m\in\mathbb{Z}^2} \left|\alpha_m^{\pm}\right|^2 \left(1 + \left|\lambda_m^{\pm}\right|^4\right) < \infty \right\} \tag{11}$$

with the inner product

$$(U, V)_Z = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} \overline{\beta_m^{\pm}} \left(1 + |\lambda_m^{\pm}|^4\right)$$

for $U = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm}$ and $V = \sum_{m\in\mathbb{Z}^2} \beta_m^{\pm} U_m^{\pm}$.

Lemma 3.3. *Z is a Hilbert space.*

Proof. Clearly Z is an inner product space so all we need to check is completeness. Thus let

$$U_n = \sum_{m\in\mathbb{Z}^2} \alpha_{m,n}^{\pm} U_m^{\pm}$$

be a Cauchy sequence in Z then

$$\sum_{m\in\mathbb{Z}^2} \left|\alpha_{m,n}^{\pm} - \alpha_{m,k}^{\pm}\right|^2 \left(1 + |\lambda_m^{\pm}|^4\right) \to 0 \text{ as } n, k \to \infty.$$

Thus

$$\sum_{m\in\mathbb{Z}^2} \alpha_{m,n}^{\pm} \sqrt{1 + |\lambda_m^{\pm}|^4}$$

is a Cauchy sequence in $\ell^2(\mathbb{C})$ and hence converges i.e.

$$\sum_{m\in\mathbb{Z}^2} \alpha_{m,n}^{\pm} \sqrt{1 + |\lambda_m^{\pm}|^4} \to \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} \sqrt{1 + |\lambda_m^{\pm}|^4} \text{ in } \ell^2(\mathbb{C}) \text{ as } n \to \infty.$$

Thus it follows that $U_n \to U = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm}$ in Z as $n \to \infty$; hence Z is a Hilbert space. $\square$

Remark 3.4. From the definition of the space Z it follows that the set

$$S = \mathrm{Span}_{\mathbb{C}} \left\{U_m^{\pm} : m \in \mathbb{Z}^2\right\} \cap Z,$$

will be a dense subset of Z and the set of eigenfunction $\left\{U_m^{\pm} : m \in \mathbb{Z}^2\right\}$ will be a Hilbert basis for the complexification of Z which we denote by $Z_{\mathbb{C}} := Z \oplus \mathrm{i}Z$.

Remark 3.5. Later we will show, in the proof of lemma 3.10, that there exists c and $C > 0$ such that

$$c(1 + |m|^{\frac{1}{2}}) \leqslant \left|\lambda_m^{\pm}\right| \leqslant C\left(1 + |m|\right),$$

for all $m \in \left\{n \in \mathbb{Z}^2 : \eta^T A \eta \neq n^T A n\right\}$. Thus the growth of the eigenvalues $\lambda_m^{\pm}$ are bounded above and below in terms of m.

Now if we take $U = (u_1, u_2) \in Z$ and we rewrite each of its components in terms of the Fourier basis $\left\{\exp\left(\mathrm{i}m \cdot \xi\right) : m \in \mathbb{Z}^2\right\}$, for the periodic Sobolev spaces $H^s(T^2)$, then we can use the above growth rates to show that

$$Z \hookrightarrow H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right),$$

is a continuous embedding.

On the other hand if we have $V \in H^2\left(T^2\right) \times H^1\left(T^2\right)$ and we write V in terms of the eigenfunctions $U_m^{\pm}$ we see that

$$H^2\left(T^2\right) \times H^1\left(T^2\right) \hookrightarrow Z$$

is a continuous embedding.

Next we define the space X which $\mathcal{A}$ maps into Z. The natural choice for this space would be the domain of $\mathcal{A}$ with the graph norm, but for this to be a Banach space we need to check that $\mathcal{A}$ is a closed operator on Z.

Lemma 3.6. *$\mathcal{A}$ is a closed operator on Z.*

Proof. The idea of this proof is to show that $\mathcal{A}$ is a closed operator on $H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$ and then use the continuous embedding of Z into $H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$ to show closedness on Z.
To avoid confusion we denote the extended operator by $\hat{\mathcal{A}} : \mathrm{D}(\hat{\mathcal{A}}) \to H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$, where $\mathrm{D}(\hat{\mathcal{A}}) := \left\{U \in H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right) : \mathcal{A}U \in H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)\right\}$.
Let $\hat{U}_n = (\hat{u}_n^1, \hat{u}_n^2) \in \mathrm{D}(\hat{\mathcal{A}})$ be a sequence such that $\hat{U}_n \to \hat{U} = (\hat{u}^1, \hat{u}^2)$ and $\hat{\mathcal{A}}\hat{U}_n = (\hat{v}_n^1, \hat{v}_n^2) \to \hat{V} = (\hat{v}^1, \hat{v}^2)$ in $H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$ as $n \to \infty$. Then

$$\mathcal{A}\hat{U}_n = \begin{pmatrix} \hat{u}_n^2 \\ -\frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi\left(A\nabla_\xi \hat{u}_n^1\right) + (\eta^T A\eta)\hat{u}_n^1\right) - c\hat{u}_n^2 - \frac{2}{\varepsilon}k^T A\nabla_\xi \hat{u}_n^2 \end{pmatrix};$$

so $\hat{v}_n^1 = \hat{u}_n^2 \to \hat{u}^2$ as $n \to \infty$ in $H^1(T^2)$ and for $\phi \in C^\infty\left(T^2\right)$ we have

$$\begin{array}{ll} \langle \hat{v}_n^2, \phi\rangle = & \langle \hat{u}_n^1, -\frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi\left(A\nabla_\xi\phi\right) + \left(\eta^T A\eta\right)\phi\right)\rangle + \langle \hat{u}_n^2, -c\phi + \frac{2}{\varepsilon}k^T A\nabla_\xi\phi\rangle \\ \downarrow & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \langle \hat{v}^2, \phi\rangle & \langle \hat{u}^1, -\frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi\left(A\nabla_\xi\phi\right) + \left(\eta^T A\eta\right)\phi\right)\rangle + \langle \hat{u}^2, -c\phi + \frac{2}{\varepsilon}k^T A\nabla_\xi\phi\rangle \end{array}$$

as $n \to \infty$ by Hölder's inequality. Hence

$$\hat{v}^2 = -\frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi\left(A\nabla_\xi \hat{u}^1\right) + \left(\eta^T A\eta\right)\hat{u}^1\right) - c\hat{u}^2 - \frac{2}{\varepsilon}k^T A\nabla_\xi \hat{u}^2$$

in the sense of distributions and it follows that

$$\hat{\mathcal{A}}\hat{U} = \hat{V} \text{ and } \hat{U} \in \mathrm{D}(\hat{\mathcal{A}});$$

thus $\hat{\mathcal{A}}$ is a closed operator.
Now we want to show that $\mathcal{A}$ is a closed operator. Let $U_n = \left(u_n^1, u_n^2\right) \in \mathrm{D}(\mathcal{A}) := \{U \in Z : \mathcal{A}U \in Z\}$ be a sequence such that $U_n \to U = (u^1, u^2)$ and $\mathcal{A}U_n = \left(v_n^1, v_n^2\right) \to V = (v^1, v^2)$ in Z as $n \to \infty$. Then, by the continuous embedding of Z into $H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$, these convergences hold in $H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$ and since $\hat{\mathcal{A}}$ is an extension of $\mathcal{A}$ we have $\hat{\mathcal{A}}U = V$; which implies $\mathcal{A}U = V$ and $U \in \mathrm{D}(\mathcal{A})$. Hence $\mathcal{A}$ is a closed operator on Z. ☐

Thus we take $X = \mathrm{D}(\mathcal{A})$ with the graph norm

$$\|U\|_X = \|U\|_Z + \|\mathcal{A}U\|_X .$$

In actual fact it is possible to characterise X in an alternative way which fits in nicely with how we defined Z;

$$X = \left\{U = \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm U_m^\pm : \alpha_m^\pm \in \mathbb{C}, \overline{\alpha_m^\pm} = \alpha_{-m}^\pm \text{ and } \sum_{m\in\mathbb{Z}^2} \left|\alpha_m^\pm\right|^2\left(1 + \left|\lambda_m^\pm\right|^6\right) < \infty\right\} \tag{12}$$

with the inner product

$$(U, V)_X = \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm \overline{\beta_m^\pm}\left(1 + \left|\lambda_m^\pm\right|^6\right)$$

for $U = \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm U_m^\pm$ and $V = \sum_{m\in\mathbb{Z}^2} \beta_m^\pm U_m^\pm$.

Remark 3.7. Similarly to Remark 3.5 there exist continuous embeddings

$$H^3\left(T^2\right) \times H^2\left(T^2\right) \hookrightarrow X \hookrightarrow H^2\left(T^2\right) \times H^{\frac{3}{2}}\left(T^2\right).$$

3.4. **Properties of $\mathcal{A}$ and F.** Now that we have defined the spaces we are working on we are now in a position to check the first hypothesis (H1).

Lemma 3.8. *$\mathcal{A} \in \mathcal{L}(X, Z)$ and $F \in C^2(X \times \mathbb{R}, X)$.*

Proof. Firstly for $U \in X$

$$\|\mathcal{A}U\|_Z \leqslant \|U\|_Z + \|\mathcal{A}U\|_Z = \|U\|_X\,;$$

so $\mathcal{A} \in \mathcal{L}(X, Z)$. Now to show $F \in C^2(X \times \mathbb{R}, X)$ we first need to show that it makes sense as a map from $X \times \mathbb{R}$ into X; thus we want to show

$$F(U, \delta) = F((u_1, u_2), \delta) = \begin{pmatrix} 0 \\ -\delta u_1 - pq(u_1) \end{pmatrix} \in X$$

for all $(U, \delta) \in X \times \mathbb{R}$. Let $(U, \delta) \in X \times \mathbb{R}$ be arbitrary then, by the embeddings in remark 3.7, $U \in H^2(T^2) \times H^{\frac{3}{2}}(T^2)$ and therefore, since $u_1 \in H^2(T^2)$ and $q \in C^\infty(\mathbb{R})$, it follows from [23, Proposition 13.3.9] that $q(u_1) \in H^2(T^2)$. Thus, as $H^2(T^2)$ is an algebra, $p \in H^2(T^2)$ and $\delta \in \mathbb{R}$, we have that $-\delta u_1 - pq(u_1) \in H^2(T^2)$. Finally since $0 \in H^3(T^2)$ we get that

$$\begin{pmatrix} 0 \\ -\delta u_1 - pq(u_1) \end{pmatrix} \in H^3(T^2) \times H^2(T^2) \subset X.$$

Thus $F : X \times \mathbb{R} \to X$ is a well-defined map.
Now to prove the required regularity of F we can just differentiate. For $((h_1, h_2), h_3)$ and $((g_1, g_2), g_3) \in X \times \mathbb{R}$ we have

$$D_{(U,\delta)}F(U, \delta)(h_1, h_2, h_3) = \begin{pmatrix} 0 & 0 & 0 \\ -\delta - pq'(u_1) & 0 & 0 \\ 0 & 0 & -u_1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

and

$$D^2_{(U,\delta)}F(U, \delta)(h_1, h_2, h_3)(g_1, g_2, g_3) = \begin{pmatrix} 0 \\ -pq''(u_1)h_1 g_1 \\ -g_1 h_3 \end{pmatrix}.$$

Therefore, since the second derivative is continuous on $X \times \mathbb{R}$, we have the desired regularity. ☐

Thus with the observation that $F(0, 0) = 0$ and $D_{(U,\delta)}F(0, 0) = 0$ we have checked hypothesis (H1) with $k = 2$. Next we will check hypothesis (H2) for this we need to prove some properties of the spectrum and define projections onto different parts of the space.

3.5. **Spectral Properties of $\mathcal{A}$.** We start by proving a growth property for the real parts of the eigenvalues $\lambda_m^\pm$ which we calculated earlier with respect to m. This growth will allow us to prove that the spectrum of $\mathcal{A}$ has spectral gaps either side of the imaginary axis, consists of just the eigenvalues $\lambda_m^\pm$ for $m \in \mathbb{Z}^2$ and has only a finite number of eigenvalues on the imaginary axis.

Remark 3.9. As we are dealing with the spectrum of $\mathcal{A}$ we need to consider $\mathcal{A}$ acting on the complexification of the Banach space Z which we will denote by $Z_\mathbb{C} := Z \oplus \mathrm{i}Z$.

Lemma 3.10. *$|Re\lambda_m^\pm| \to \infty$ as $|m| \to \infty$.*

Proof. Let $S_A^1 = \{k \in \mathbb{R}^2 k^T A k = 1\}$. For $k \in S_A^1$ there exists a $k_\perp \in S_A^1$ such that $\{k, k_\perp\}$ is a orthonormal basis for $\mathbb{R}^2$ with respect to the inner product $(x, y)_A = x^T A y$. Thus for any $m \in \mathbb{Z}^2$ we have

$$m = (m, k)_A k + (m, k_\perp)_A k_\perp;$$

which implies that

$$m^T A m = (m, k)_A^2 + (m, k_\perp)_A^2\,.$$

Now, using this decomposition of m and the expression for $m^T A m$ in the formula for $\lambda_m^\pm$ (9), we get that

$$\mathrm{Re}\lambda_m^\pm = \frac{1}{2}\left(-c \pm \mathrm{Re}\sqrt{c^2 + \frac{4}{\varepsilon^2}(m, k_\perp)_A^2 + \frac{4c\mathrm{i}}{\varepsilon}(m, k)_A - \frac{4}{\varepsilon^2}\eta^T A \eta}\right).$$

Hence, as

$$
\begin{aligned}
&\mathrm{Re}\sqrt{c^2+\frac{4}{\varepsilon^2}(m,k_\perp)_A^2+\frac{4c\mathrm{i}}{\varepsilon}(m,k)_A-\frac{4}{\varepsilon^2}\eta^TA\eta}\\
&=\frac{1}{\sqrt{2}}\sqrt{\left|c^2+\frac{4}{\varepsilon^2}\left((m,k_\perp)_A^2-\eta^TA\eta\right)+\frac{4c\mathrm{i}}{\varepsilon}(m,k)_A\right|+c^2+\frac{4}{\varepsilon^2}\left((m,k_\perp)_A^2-\eta^TA\eta\right)};
\end{aligned}
$$

which tends to ∞ as $|m|\to\infty$, it follows that $|\mathrm{Re}\lambda_m^\pm|\to\infty$ as $|m|\to\infty$. □

Remark 3.11. The preceding lemma also gives upper and lower bounds on the growth of the real part of the eigenvalues with respect to m, since the growth will only depend on the growth of the square root. Thus there exist $c, C>0$ such that for all $m\in\left\{n\in\mathbb{Z}^2:\eta^TA\eta\neq n^TAn\right\}$

$$
c(1+|m|^{\frac{1}{2}})\leqslant\left|\mathrm{Re}\lambda_m^\pm\right|\leqslant C(1+|m|). \tag{13}
$$

Next we show that the spectrum has a gap either side of the imaginary axis by showing that the spectrum is equal to the set $\left\{\lambda_m^\pm:m\in\mathbb{Z}^2\right\}$ and then deducing that this set has gaps either side of the imaginary axis.

Lemma 3.12. *$\sigma(\mathcal{A})=\left\{\lambda_m^\pm:m\in\mathbb{Z}^2\right\}$ and $\sigma(\mathcal{A})$ has spectral gaps either side of the imaginary axis.*

Proof. This result is proved by showing that any point not in $\left\{\lambda_m^\pm:m\in\mathbb{Z}^2\right\}$ is a resolvent point. Let $\lambda\in\mathbb{C}\setminus\left\{\lambda_m^\pm:m\in\mathbb{Z}^2\right\}$ be arbitrary then since $|\mathrm{Re}\lambda_m^\pm|\to\infty$ as $|m|\to\infty$ we have that

$$
\rho=\inf_{\mu\in\left\{\lambda_m^\pm:m\in\mathbb{Z}^2\right\}}|\lambda-\mu|>0.
$$

Now the set of eigenfunctions $\left\{U_m^\pm:m\in\mathbb{Z}^2\right\}$ is a Hilbert basis for $Z_{\mathbb{C}}$, so $\mathcal{S}_{\mathbb{C}}:=\mathrm{Span}_{\mathbb{C}}\left\{U_m^\pm:m\in\mathbb{Z}\right\}$ is a dense subset of $Z_{\mathbb{C}}$. Hence, as $\mathcal{A}$ is a closed operator, to show that $(\mathcal{A}-\lambda I)$ has a bounded inverse on $Z_{\mathbb{C}}$ we just need to show that one exists on $\mathcal{S}_{\mathbb{C}}$.
Let $V=\sum_{m\in\mathbb{Z}^2}\alpha_m^\pm U_m^\pm\in\mathcal{S}_{\mathbb{C}}$; then we can define

$$
U=\sum_{m\in\mathbb{Z}^2}\frac{\alpha_m^\pm}{\lambda_m^\pm-\lambda}U_m^\pm\in\mathcal{S}_{\mathbb{C}} \tag{14}
$$

such that $(\mathcal{A}-\lambda I)\,U=V$ and

$$
\|U\|_Z\leqslant\max_{\mu\in\left\{\lambda_m^\pm:m\in\mathbb{Z}^2\right\}}\left\{\frac{1}{|\mu-\lambda|}\right\}\|V\|_Z\leqslant\frac{1}{\rho}\|V\|_Z\,.
$$

Thus we have a bounded inverse defined on $\mathcal{S}_{\mathbb{C}}$ and λ is a resolvent point. Hence $\sigma(\mathcal{A})=\left\{\lambda_m^\pm:m\in\mathbb{Z}^2\right\}$ and furthermore, since $|\mathrm{Re}\lambda_m^\pm|\to\infty$ as $m\to\infty$, there can only be a finite number of eigenvalues near the imaginary axis. Thus there is a gap in the spectrum either side of the imaginary axis. □

We have now shown that $\mathcal{A}$ has a spectral gap either side of the imaginary axis. The next step is to characterise the eigenvalues which lie on the imaginary axis.

Lemma 3.13. *An eigenvalue λ has zero real part if and only if $\lambda=\lambda_m^+$ for $m\in\mathbb{Z}$ such that*

$$
m^TAm=\eta^TA\eta;
$$

where η is the η chosen in theorem 2.1. Furthermore zero is the only eigenvalue with zero real part.

Proof. Suppose $\lambda=\lambda_m^+$ for $m^TAm=\eta^TA\eta$ then

$$
\lambda_m^+=\frac{-\left(c+\frac{2\mathrm{i}}{\varepsilon}k^TAm\right)+\sqrt{\left(c+\frac{2\mathrm{i}}{\varepsilon}k^TAm\right)^2}}{2}=0
$$

which has zero real part.

On the other hand suppose an eigenvalue λ has zero real part then λ solves

$$\lambda^2 + \left(c + \frac{2\mathrm{i}}{\varepsilon} k^T A l\right) \lambda + \frac{1}{\varepsilon^2} \left(\eta^T A \eta - l^T A l\right) = 0$$

for some $l \in \mathbb{Z}^2$. Let $\tilde{\lambda}$ be the other root of this quadratic equation then by the properties of roots we have

$$\lambda \tilde{\lambda} = \frac{1}{\varepsilon^2} \left(\eta^T A \eta - l^T A l\right)$$
$$\lambda + \tilde{\lambda} = -\left(c + \frac{2\mathrm{i}}{\varepsilon} k^T A m\right).$$

Thus, since the real part of λ is zero, the first equations implies that either $\mathrm{Re}\tilde{\lambda} = 0$ or $\mathrm{Im}\lambda = 0$. However, as $c \neq 0$, the second equation tells use that $\mathrm{Re}\tilde{\lambda} \neq 0$; hence $\mathrm{Im}\lambda = 0$ and it follows that $\lambda = 0$. Then from rearranging the first equation we get $\eta^T A \eta = l^T A l$. Finally, since

$$\lambda_l^- = -\left(c + \frac{2\mathrm{i}}{\varepsilon} k^T A l\right) \neq 0$$

we have $\lambda = \lambda_l^+$.
Furthermore from these two calculations it follows that any eigenvalue with zero real part will be zero. □

Remark 3.14. The proof, of the above lemma, tells use that the number of eigenvectors associated with the zero eigenvalue will be

$$\chi(\eta) = \# \left\{ m \in \mathbb{Z}^2 : \eta^T A \eta = m^T A m \right\}.$$

Thus the dimension of X_c will be $\chi(\eta)$.

The next step is to define the centre space X_c. Before doing this we introduce

$$\begin{aligned} \mathcal{S} &= \left\{ U_m^\pm | m \in \mathbb{Z}^2 \right\}, & S &= \mathrm{span}_{\mathbb{C}} \left\{ \mathcal{S} \right\} \cap Z, \\ \mathcal{S}^s &= \left\{ U_m^\pm | \mathrm{Re}\lambda_m^\pm < 0 \right\}, & S^s &= \mathrm{span}_{\mathbb{C}} \left\{ \mathcal{S}^s \right\} \cap Z, \\ \mathcal{S}^u &= \left\{ U_m^\pm | \mathrm{Re}\lambda_m^\pm > 0 \right\}, & S^u &= \mathrm{span}_{\mathbb{C}} \left\{ \mathcal{S}^u \right\} \cap Z, \\ \mathcal{S}^c &= \left\{ U_m^\pm | \mathrm{Re}\lambda_m^\pm = 0 \right\}, & S^c &= \mathrm{span}_{\mathbb{C}} \left\{ \mathcal{S}^c \right\} \cap Z; \end{aligned}$$

the last three lines correspond to the sets and spans of the stable, unstable and centre eigenfunctions. Thus we define

$$X_c = Z_c := S^c, \tag{15}$$

which is a finite dimensional space by the previous lemma. Hence we have defined a finite dimensional space for which the restriction of the operator $\mathcal{A}$ to it has spectrum on the imaginary axis.
Thus to finish checking hypothesis (H2) we need to define a projection $\pi_c \in \mathcal{L}(Z, X)$ onto X_c and check that it commutes with $\mathcal{A}$. π_c is defined in the following way: let $\tilde{\pi}_c$ be the projection of S onto S_c, defined in the natural way, then for each $U \in Z$ define $\pi_c : Z \to X_c$ by

$$\pi_c U = \lim_{n \to \infty} \tilde{\pi}_c U_n;$$

where $U_n \in S$ is a sequence such that $U_n \to U$ in Z as $n \to \infty$.

Lemma 3.15. *π_c is well-defined and $\pi_c \in \mathcal{L}(Z, X)$.*

Proof. Let $U \in Z$ then, since S is a dense subset of Z, there exist a sequence $U_n \in S$ such that $U_n \to U$ in Z as $n \to \infty$. Now, since $\mathcal{A}\tilde{\pi}_c U_n = 0$ for all $n \in \mathbb{N}$, we have

$$\left\| \tilde{\pi}_c U_n - \tilde{\pi}_c U_m \right\|_X = \left\| \tilde{\pi}_c U_n - \tilde{\pi}_c U_m \right\|_Z$$

and it follows that $\tilde{\pi}_c U_n$ is a Cauchy sequence in X_c. Thus, as X_c is finite dimensional, $\tilde{\pi}_c U_n \to \pi_c U$ in X_c as $n \to \infty$. This limit is independent of the choice of sequence by an interlacing argument; hence π_c is well defined. Finally

$$\|\pi_c U\|_X = \|\pi_c U\|_Z \text{ for all } U \in Z;$$

so $\pi_c \in \mathcal{L}(Z, X)$. □

Furthermore if we define Z_s, Z_u and Z_h to be the closures of S^s, S^u and $S^h := \mathrm{span}_{\mathbb{C}}\{\mathcal{S}^u \cup \mathcal{S}^s\} \cap Z$ in Z; then we can define projections π_s, π_u and $\pi_h \in \mathcal{L}(Z)$ and $\mathcal{L}(X)$ onto these spaces in a similar way. Hence we can decompose Z as follows

$$Z = Z_s \oplus X_c \oplus Z_u = X_c \oplus Z_h.$$

Finally, to complete the verification of (H2), it just remains to show that π_c commutes with $\mathcal{A}$.

Lemma 3.16. *π_c, π_s and π_u commute with $\mathcal{A}$.*

Proof. This result is proved in two steps; first we work out exactly how $\mathcal{A}$ acts on an element of X and then we use this to show that the projections commute with $\mathcal{A}$.
Let $U = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm} \in X$ then the partial sums $U_n = \sum_{|m|\leqslant n} \alpha_m^{\pm} U_m^{\pm} \to U$ in X as $n \to \infty$ and

$$\mathcal{A}U_n = \sum_{|m|\leqslant n} \lambda_m^{\pm}\alpha_m^{\pm}U_m^{\pm} \to \sum_{m\in\mathbb{Z}^2} \lambda_m^{\pm}\alpha_m^{\pm}U_m^{\pm}$$

in Z as $n \to \infty$. Therefore, as $\mathcal{A}$ is a closed operator, it follows that

$$\mathcal{A}U = \sum_{m\in\mathbb{Z}^2} \lambda_m^{\pm}\alpha_m^{\pm}U_m^{\pm}.$$

Now if we let $U = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm} \in X$ be arbitrary then we have that

$$\pi_c\mathcal{A}U = \pi_c \sum_{m\in\mathbb{Z}^2} \lambda_m^{\pm}\alpha_m^{\pm}U_m^{\pm} = \sum_{m\in\mathbb{Z}^2} \lambda_m^{\pm}\alpha_m^{\pm}\pi_c U_m^{\pm} = \mathcal{A}\pi_c U;$$

so $\mathcal{A}$ commutes with π_c. A similar argument works for π_s and π_u. □

Hence we have completed the verification of hypothesis (H2) all that remains now is to check hypothesis (H3). To do this we first need to construct exponentially decaying semigroups on Z_s and Z_u.

3.6. **Exponentially Decaying Semigroups.**

Definition 3.17. *Let $T(t)$ be a C_0-semigroup on a Banach space E then we say T is an exponentially decaying semigroup if there exist γ and $C \geqslant 1$ such that*

$$\|T(t)\|_{\mathcal{L}(E)} \leqslant Ce^{-\gamma t}$$

for all $t \geqslant 0$. In this case we call γ the decay constant of the semigroup.

Remark 3.18. The semigroups we construct are C_0-semigroups but not analytic because the operator $\mathcal{A}$ is not bisectorial.

Lemma 3.19. *Let $X_s = D(\mathcal{A}) \cap Z_s$, $X_u = D(\mathcal{A}) \cap Z_u$, $\mathcal{A}_s = \mathcal{A}|_{X_s}$ and $\mathcal{A}_u = \mathcal{A}|_{X_u}$; then $\mathcal{A}_s$ and $-\mathcal{A}_u$ are closed operators which generate exponentially decaying C_0-semigroups of contractions on Z_s and Z_u respectively.*

Proof. We will prove this result for $\mathcal{A}_s$, an almost identical argument can be used for $-\mathcal{A}_u$.
We prove this result in two steps, first we show that $\mathcal{A}_s$ generates a C_0-semigroup of contractions; then we show that this semigroup is exponentially decaying.
To show that $\mathcal{A}_s$ generates a C_0-semigroup of contractions we use the Hille-Yoshida Theorem, which can be found in the book by Pazy [18, Theorem 1.3.1], thus we need to show

(1) $\mathcal{A}_s$ is closed and $\overline{X_s} = Z_s$,

(2) $\rho(\mathcal{A}_s) \supset \mathbb{R}^+$ and for every $\lambda > 0$ we have

$$\left\|(\mathcal{A}_s - \lambda I)^{-1}\right\| \leqslant \frac{1}{\lambda}.$$

Now for the first part $S^s \subset X_s$ so, as S^s is dense in Z^s, it follows that $\overline{X_s} = Z_s$. Furthermore the closedness of $\mathcal{A}_s$ follows from the closedness of $\mathcal{A}$; let $U_n \in X_s$ be a sequence such that $U_n \to U$ and $\mathcal{A}_s U_n \to V$ in Z as $n \to \infty$. Then, since $\mathcal{A}$ is closed, $\mathcal{A}U = V$ and $U \in \mathrm{D}(\mathcal{A})$ and, since $U_n \to U$ and Z_s is closed, we have that $U \in Z_s$. Hence $U \in X_s$ and $\mathcal{A}_s U = \mathcal{A}U = V$; so $\mathcal{A}_s$ is closed.
Now we prove the properties of the resolvent set for part 2. Let $\lambda \geqslant 0$ and $V = \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm U_m^\pm \in S^s$, then if we take

$$U = \sum_{m\in\mathbb{Z}^2} \frac{\alpha_m^\pm}{\lambda_m^\pm - \lambda} U_m^\pm \in S^s;$$

we have that $(\mathcal{A} - \lambda I)U = V$ and

$$\left\|U\right\|_z \leqslant \frac{1}{\gamma + \lambda} \left\|V\right\|_Z , \tag{16}$$

where $\gamma = \min\left\{|\mathrm{Re}\lambda_m^\pm| : \mathrm{Re}\lambda_m^\pm \neq 0\right\} > 0$. Thus we have a bounded densely defined inverse and so, as $\mathcal{A}_s$ is a closed operator, we get that $\lambda \in \rho(\mathcal{A}_s)$ for all $\lambda \geqslant 0$ and we can extend this inverse and estimate to the whole of Z_s, to obtain our desired estimate for $\lambda > 0$. Hence by the Hille-Yoshida Theorem there exists a C_0-semigroup of contractions on Z_s, which we will denote by $T_s(\tau)$.
To complete the proof it just remains to show that $T_s(\tau)$ is an exponentially decaying semigroup. We prove this by using the spectrum of $T_s(\tau)$ to bound its norm.
The pairs $\left(\mathrm{e}^{\lambda_m^\pm \tau}, U_m^\pm\right)$ are eigenpairs of $T_s(\tau)$ and, since the eigenfunctions $U_m^\pm$ form a Hilbert basis for the complexification of Z_s, we can use a similar argument to the one used to show that $\rho(\mathcal{A}) = \left\{\lambda_m^\pm : m \in \mathbb{Z}^2\right\}$ to deduce that these are all the whole spectrum of $T_s(\tau)$.
Now if we choose $U = \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm U_m^\pm \in S^s$ arbitrarily then

$$\left\|T_s(\tau)U\right\|_Z \leqslant \mathrm{e}^{-\gamma\tau} \left\|U\right\|_Z$$

and this estimate can be extended to the whole of Z_s since S^s is dense in Z_s and $T_s(\tau)$ is a bounded linear operator. Thus $T_s(\tau)$ is an exponentially decaying semigroup. □

Notation 3.20. The C_0-semigroups generated by $\mathcal{A}_s$ and $-\mathcal{A}_u$ will be denoted by $T_s(\tau)$ and $T_u(\tau)$, and their common decay constant will be denoted by

$$\gamma = \inf\left\{\left|\mathrm{Re}\lambda_m^\pm\right| : \mathrm{Re}\lambda_m^\pm \neq 0\right\}.$$

Now that we have constructed exponentially decaying semigroups on the the stable and unstable parts of the space we are in a position to check that the affine problem on the hyperbolic part of the space, introduced in hypothesis (H3), has a solution satisfying the necessary bound.

3.7. **Affine Problem.** In this section we construct solutions to the affine problem on the hyperbolic part of the space. This requires a bit of work as the operator $\mathcal{A}$ is not a bisectorial operator, so we do not have analytic semigoups.

Lemma 3.21. *Let $X_h := X_s \oplus X_u$ then for each $\eta \in [0, \gamma)$ and $f \in C_\eta(\mathbb{R}, X_h)$ the space of continuous functions with exponential growth of rate η, the affine problem*

$$U_\tau^h = \mathcal{A}U^h + f \text{ and } U^h \in C_\eta(\mathbb{R}, X_h) \tag{17}$$

has a unique solution $U^h = K_h f$, where $K_h \in \mathcal{L}\left(C_\eta(\mathbb{R}, X_h)\right)$ and

$$\left\|K_h\right\|_{\mathcal{L}(C_\eta(\mathbb{R},X_h))} \leqslant \Gamma(\eta);$$

for some continuous function $\Gamma : [0, \gamma) \to \mathbb{R}^+$.

Proof. We construct a solution to the above affine problem by combining solutions to the problem when we restrict it to X_s and X_u.
Thus we start by considering the affine problem restricted to X_s, for $\eta \in [0,\gamma)$ and $f_s \in C_\eta(\mathbb{R}, X_s)$ find $U^s \in C_\eta(\mathbb{R}, X_s)$ such that

$$U^s_\tau = \mathcal{A}_s U + f_s. \tag{18}$$

Now as $\mathcal{A}_s$ generates an exponentially decaying C_0-semigroup of contractions we can define

$$U^s(\tau) = \int_{-\infty}^{\tau} T_s(\tau-\sigma) f_s(\sigma) \mathrm{d}\sigma,$$

we claim that this function solves the affine problem restricted to X_s (18).
Claim: $U^s \in C^1(\mathbb{R}, Z_s) \cap C_\eta(\mathbb{R}, X_s)$ and solves (18).

Proof of Claim. The first step is to check that U_s has the required regularity. We begin by proving $U^s \in C(\mathbb{R}, Z_s)$. Let $\tau \in \mathbb{R}$ be arbitrary and $h \in (-1,1)$ then

$$\begin{aligned}
\|U^s(\tau+h) - U^s(\tau)\|_Z &= \left\| \int_{-\infty}^{\tau+h} T_s(\tau+h-\sigma) f_s(\sigma)\mathrm{d}\sigma - \int_{-\infty}^{\tau} T_s(\tau-\sigma) f_s(\sigma)\mathrm{d}\sigma \right\|_Z \\
&\leqslant \left\| \int_{-\infty}^{\tau} T_s(\tau-\sigma)\left(f_s(\sigma+h) - f_s(\sigma)\right)\mathrm{d}\sigma \right\|_Z \\
&\leqslant \left| \int_{-\infty}^{\tau} \|T_s(\tau-\sigma)\|_{\mathcal{L}(Z)} \|f_s(\sigma+h) - f_s(\sigma)\|_Z \right| \mathrm{d}\sigma.
\end{aligned}$$

Now, since $f_s \in C_\eta(\mathbb{R}, X_s)$ and T_s is exponentially decaying, for $k < \min\{0,\tau\}$ we have the estimate

$$\begin{aligned}
&\left| \int_{-\infty}^{k} \|T_s(\tau-\sigma)\|_{\mathcal{L}(Z)} \|f_s(\sigma+h) - f_s(\sigma)\mathrm{d}\sigma\|_Z \right| \\
&\qquad \leqslant \|f_s\|_\eta \int_{-\infty}^{k} \mathrm{e}^{-\gamma(\tau-\sigma)} \left(\mathrm{e}^{\eta|\sigma+h|} + \mathrm{e}^{\eta|\sigma|}\right) \mathrm{d}\sigma \\
&\qquad \leqslant 2\|f_s\|_\eta \mathrm{e}^{-\tau\gamma+\eta} \int_{-\infty}^{k} \mathrm{e}^{(\gamma-\eta)\sigma}\mathrm{d}\sigma \\
&\qquad \leqslant \frac{2}{\gamma-\eta}\|f_s\|_\eta \mathrm{e}^{-\tau\gamma+\eta}\mathrm{e}^{(\gamma-\eta)k} \to 0
\end{aligned}$$

as $k \to -\infty$. Thus if we let $\varepsilon > 0$ be arbitrary then there exists a $k < \min\{0,\tau\}$ such that

$$\left| \int_{-\infty}^{k} \|T_s(\tau-\sigma)\|_{\mathcal{L}(Z)} \|f_s(\sigma+h) - f_s(\sigma)\|_Z \mathrm{d}\sigma \right| < \frac{\varepsilon}{2}.$$

Now for this fixed k we know that f_s is uniformly continuous on $[k,\tau]$ so there exist a $\delta > 0$ such that if $|h| < \delta$

$$\left| \int_{k}^{\tau} \|T_s(\tau-\sigma)\|_{\mathcal{L}(Z)} \|f_s(\sigma+h) - f_s(\sigma)\|_Z \mathrm{d}\sigma \right| < \frac{\varepsilon}{2}.$$

Combining these two estimates we conclude that $\|U_s(\tau+h) - U_s(\tau)\| \to 0$ as $h \to 0$ and thus $U^s \in C(\mathbb{R}, Z_s)$.
The next step is to show that $U^s \in C(\mathbb{R}, X_s)$. In order to do this we first calculate $\mathcal{A}_s U^s$. Thus let $\tau \in \mathbb{R}$ be arbitrary then we can approximate $U^s(\tau)$ by the integrals

$$U^s_k(\tau) = \int_{k}^{\tau} T_s(\tau-\sigma) f_s(\sigma)\mathrm{d}\sigma,$$

for $k \leqslant \tau$ which will converge to $U^s(\tau)$ as $k \to -\infty$. Thus, since $\mathcal{A}_s$ is a closed operator,

$$\mathcal{A}_s U^s_k(\tau) = \int_{k}^{\tau} T_s(\tau-\sigma) \mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma$$

and this integral converges to

$$\int_{-\infty}^{\tau} T_s(\tau-\sigma)\mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma$$

as $k \to -\infty$. Therefore, as $\mathcal{A}$ is a closed operator, it follows that

$$\mathcal{A}_s U^s = \int_{-\infty}^{\tau} T_s(\tau-\sigma)\mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma.$$

Now we are in a position to prove continuity in X_s. Let $h \in (-1,1)$ then, since X_s is equipped with the graph norm and $U^s \in C(\mathbb{R}, Z_s)$, all we need to prove is that $\|\mathcal{A}_s U^s(\tau+h) - \mathcal{A}_s U^s(\tau)\|_Z \to 0$ as $h \to 0$ for all $\tau \in \mathbb{R}$. Thus we let if $\tau \in \mathbb{R}$ be arbitrary then we have that,

$$\begin{aligned}
&\|\mathcal{A}_s U^s(\tau+h) - \mathcal{A}_s U^s(\tau)\| \\
&\qquad = \left\| \int_{-\infty}^{\tau+h} T_s(\tau+h-\sigma)\mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma - \int_{-\infty}^{\tau} T_s(\tau-\sigma)\mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma \right\|_Z \\
&\qquad \leqslant \left\| \int_{-\infty}^{\tau} T_s(\tau-\sigma)\left(\mathcal{A}_s f_s(\sigma+h) - \mathcal{A}_s f_s(\sigma)\right)\mathrm{d}\sigma \right\|_Z \\
&\qquad \leqslant \left| \int_{-\infty}^{\tau} \|T_s(t-\sigma)\|_{\mathcal{L}(Z)} \|\mathcal{A}_s f_s(\sigma+h) - \mathcal{A}_s f_s(\sigma)\|_Z \mathrm{d}\sigma \right|;
\end{aligned}$$

which tends to zero by a similar argument to the one given in the previous step since $f_s \in C_\eta(\mathbb{R}, X_s)$.

Next we need to show that $U^s \in C^1(\mathbb{R}, Z_s)$ and satisfies (18). Let $\tau \in \mathbb{R}$ be arbitrary and $h > 0$ then

$$\begin{aligned}
\frac{U^s(\tau+h) - U^s(\tau)}{h} &= \frac{1}{h}\left(\int_{-\infty}^{\tau+h} T_s(\tau+h)f_s(\sigma)\mathrm{d}\sigma - \int_{-\infty}^{\tau} T_s(\tau)f_s(\sigma)\mathrm{d}\sigma\right) \\
&= \frac{T_s(h)-I}{h}\int_{-\infty}^{\tau} T_s(\tau)f_s(\sigma)\mathrm{d}\sigma + \frac{1}{h}\int_{\tau}^{\tau+h} T_s(\tau+h)f_s(\sigma)\mathrm{d}\sigma \\
&\to \mathcal{A}_s U^s(\tau) + f_s(\tau) \text{ as } h \searrow 0.
\end{aligned}$$

Thus the right derivative exists and is the required result. If we replace h with $-h$ then a similar argument will give the existence of the left derivative which is also the required result. Furthermore, since $\mathcal{A}_s U_s \in C(\mathbb{R}, Z_s)$ and $f_s \in C_\eta(\mathbb{R}, X_s)$, it follows that $U^s \in C^1(\mathbb{R}, Z_s)$ and satisfies (18).

Finally it just remains to check that $U^s \in C_\eta(\mathbb{R}, X_s)$, as $f_s \in C_\eta(\mathbb{R}, X_s)$ and T_s is an exponentially decaying semigroup, we have the estimate

$$\begin{aligned}
\|U^s(\tau)\|_X &= \|U^s(\tau)\|_Z + \|\mathcal{A}_s U^s(\tau)\|_Z \\
&= \left\| \int_{-\infty}^{\tau} T_s(\tau-\sigma)f_s(\sigma)\mathrm{d}\sigma \right\|_Z + \left\| \int_{-\infty}^{\tau} T_s(\tau-\sigma)\mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma \right\|_Z \\
&\leqslant \left| \int_{-\infty}^{\tau} \|T_s(\tau-\sigma)\|_{\mathcal{L}(Z)} \left(\|f_s(\sigma)\|_Z + \|\mathcal{A}_s f_s(\sigma)\|_Z\right)\mathrm{d}\sigma \right| \\
&\leqslant \|f_s\|_\eta \int_{-\infty}^{\tau} \mathrm{e}^{-\gamma(\tau-\sigma)}\mathrm{e}^{\eta|\sigma|}\mathrm{d}\sigma \\
&= \|f_s\|_\eta \int_{0}^{\infty} \mathrm{e}^{-\gamma\sigma+\eta|\tau-\sigma|}\mathrm{d}\sigma \\
&\leqslant \frac{1}{\gamma-\eta}\|f_s\|_\eta \mathrm{e}^{\eta|\tau|};
\end{aligned}$$

hence $U^s \in C_\eta(\mathbb{R}, X_s)$ and we have completed the proof of the claim. □

A similar argument to the one given above can be used for the affine problem on Z_u to show that for $f_u \in C_\eta(\mathbb{R}, X_u)$ the equation

$$U^u_\tau = -\mathcal{A}_u U^u + f_u$$

has a solution

$$U^u(\tau) = \int_{-\infty}^{\tau} T_u(\tau - \sigma) f_u(\sigma) \mathrm{d}\sigma \in C^1(\mathbb{R}, Z_u) \cap C_\eta(\mathbb{R}, X_u).$$

We will now use these solutions to construct a solution $U^h \in C_\eta(\mathbb{R}, X_h)$ for

$$U^h_\tau = \mathcal{A}U^h + f. \tag{17}$$

Since π_s and π_u commute with $\mathcal{A}$, we can split (17) into its stable and unstable parts

$$U^s_\tau + U^u_\tau = \mathcal{A}_s U^s + \mathcal{A}_u U^u + \pi_s f + \pi_u f;$$

where $U^s = \pi_s U$ and $U^u = \pi_u U$. Thus we can solve (17) by solving the equations on the stable and unstable parts separately. Hence we want to solve the pair of equations

$$U^s_\tau = \mathcal{A}_s U^s + \pi_s f \tag{19}$$

$$U^u_\tau = \mathcal{A}_u U^u + \pi_u f. \tag{20}$$

Now we have shown that (19) has a solution

$$U^s(\tau) = \int_{-\infty}^{\tau} T_s(\tau - \sigma) \pi_s f(\sigma) \mathrm{d}\sigma;$$

so we just need to construct a solution for (20). We consider the following auxiliary problem; suppose $U^u(\tau)$ solves (20) and let $V(\tau) = U^u(-\tau)$, then

$$\begin{aligned} V_\tau(\tau) = -U^u_\tau(-\tau) &= -\mathcal{A}_u U^u(-\tau) - \pi_u f(-\tau) \\ &= -\mathcal{A}_u V(\tau) - \pi_u f(-\tau). \end{aligned}$$

This equation for V has a solution

$$V(\tau) = -\int_{-\infty}^{\tau} T_u(\tau - \sigma) \pi_u f(-\sigma) \mathrm{d}\sigma;$$

which leads to a solution of (20)

$$U^u(\tau) = V(-\tau) = -\int_{\tau}^{\infty} T_u(\sigma - \tau) \pi_u f(\sigma) \mathrm{d}\sigma.$$

Now putting these solutions on the stable and unstable parts together we get a solution for (17),

$$U^h(\tau) = \int_{-\infty}^{\tau} T_s(\tau - \sigma) \pi_s f(\sigma) \mathrm{d}\sigma - \int_{\tau}^{\infty} T_u(\sigma - \tau) \pi_u f(\sigma) \mathrm{d}\sigma.$$

This solution is unique, since if we had two solutions then we could subtract them to get a solution of the homogeneous problem; however the homogeneous problem only has the zero solution as a solution in $C_\eta(\mathbb{R}, X_h)$. Hence the original two solutions must be equal.
Hence we define the map $K_h : C_\eta(\mathbb{R}, X_h) \to C_\eta(\mathbb{R}, X_h)$ by

$$K_h f(\tau) = \int_{-\infty}^{\tau} T_s(\tau - \sigma) \pi_s f(\sigma) \mathrm{d}\sigma - \int_{\tau}^{\infty} T_u(\sigma - \tau) \pi_u f(\sigma) \mathrm{d}\sigma. \tag{21}$$

K_h is a linear map since integration, T_s, T_u, π_s and π_s are all linear. Finally to complete the proof it just remains to estimate the norm of K_h. As $f \in C_\eta(\mathbb{R}, X_h)$ and since the semigroups are exponentially decaying we have the following estimate

$$\begin{aligned}\|K_h f(\tau)\|_X &\leqslant \|\pi_s K_h f(\tau)\|_X + \|\pi_u K_h(\tau)\|_X \\ &\leqslant \left|\int_{-\infty}^{\tau} \|T_s(\tau-\sigma)\|_{\mathcal{L}(Z)} \|\pi_s f(\sigma)\|_X \,\mathrm{d}\sigma\right| \\ &\quad + \left|\int_{\tau}^{\infty} \|T_u(\sigma-\tau)\|_{\mathcal{L}(Z)} \|\pi_u f(\sigma)\|_X \,\mathrm{d}\sigma\right| \\ &\leqslant \|f\|_\eta \left(\int_{-\infty}^{\tau} \mathrm{e}^{-\gamma(\tau-\sigma)+\eta|\sigma|}\mathrm{d}\sigma + \int_{\tau}^{\infty} \mathrm{e}^{-\gamma(\sigma-\tau)+\eta|\sigma|}\mathrm{d}\sigma\right) \\ &\leqslant 2\|f\|_\eta \mathrm{e}^{\eta|t|} \int_0^\infty \mathrm{e}^{(\eta-\gamma)\sigma}\mathrm{d}\sigma \\ &\leqslant \frac{2}{\gamma-\eta}\|f\|_\eta \mathrm{e}^{\eta|t|}.\end{aligned}$$

Hence it follows that

$$\|K_h f\|_\eta \leqslant \frac{2}{\gamma-\eta}\|f\|_\eta$$

and so finally we obtain

$$\|K_h\|_{\mathcal{L}(C_\eta(\mathbb{R},X_h))} \leqslant \frac{2}{\gamma-\eta}.$$

Thus we have completed the proof of the lemma 17. □

Hence we have verified hypotheses (H1) - (H3); so we can perform a local centre manifold reduction for δ fixed as it follows directly the local centre manifold theorem with the choices of X_c and π_c made in this section.

3.8. **Proof for Full System.** In the previous subsection we showed that there is a local centre manifold reduction for the restricted system. Now the aim in this subsection is to prove proposition 3.1 by extending the local centre manifold reduction for the restricted system to the whole system, which includes δ as a variable.
To achieve this we start by writing the extended system in the form

$$\begin{pmatrix} U_\tau \\ \delta_\tau \end{pmatrix} = \mathcal{B}\begin{pmatrix} U \\ \delta \end{pmatrix} + H(U,\delta);$$

where

$$\mathcal{B} := \begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X\times\mathbb{R}, Z\times\mathbb{R})$$

and

$$H(U,\delta) := \begin{pmatrix} F(U,\delta) \\ 0 \end{pmatrix} \in C^2(X\times\mathbb{R}, X\times\mathbb{R}).$$

Notice that the properties stated above of $\mathcal{B}$ and H follow from lemma 3.8, since we only extend $\mathcal{A}$ and F by 0. We can then extend the results checking the hypotheses of the centre manifold theorem for the restricted system to the full system. Thus we get the existence of a centre manifold reduction for the full system. After this it just requires some rearrangement to prove proposition 3.1. More details are given in [5] and, the book of Haragus and Iooss [11, Theorem 3.3].

4. Solutions on the Centre Manifold

In the previous section we proved that small solutions for the infinite dimensional spatial dynamical system generated by our problem can be found by finding solutions on a finite dimensional local centre manifold. So, to complete the proof of theorem 2.1, we just need to find solutions on the local centre manifold which stay within the open neighbourhood Ω of the origin given in the theorem.

Notation 4.1. Through out this section we will need to use the expansions of the function p and q that appear in the nonlinearity,

$$F\left(\left(u_1, u_2\right), \delta\right) = \begin{pmatrix} 0 \\ -\delta u_1 - pq(u_1) \end{pmatrix}.$$

Thus, since $p \in H^2\left(T^2\right)$ and $q \in C^2\left(\mathbb{R}\right)$ with $q(0) = 0$, $q'(0) = 0$ and $q''(0) \neq 0$, we can write $p(\xi) = \sum_{m \in \mathbb{Z}^2} p_m e^{im\cdot\xi}$ and $q(s) = q_2 s^2 + O(s^3)$.

4.1. **Solutions on the One Dimensional Centre Manifold for $\mu = -\delta$.** We start by looking at the case when $\chi(\eta) = 1$ and thus $\mu = -\delta$. For this case, since the equation $m^T Am = 0$ only has one solution $(0, 0) \in \mathbb{Z}^2$, it follows from remark 3.14 that X_c is 1-dimensional and

$$X_c = \begin{cases} \mathrm{Span}_{\mathbb{R}}\left\{U_0^+\right\} & \text{if } c > 0 \\ \mathrm{Span}_{\mathbb{R}}\left\{U_0^-\right\} & \text{if } c < 0 \end{cases} = \mathrm{Span}_{\mathbb{R}}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}.$$

Throughout this subsection we will assume that $c > 0$ and $X_c = \mathrm{span}_{\mathbb{R}}\left\{U_c^+\right\}$. If $c < 0$ then the calculation from this section will work if the roles of U_0^+ and U_0^- are interchanged.
We will show that for $p \in H^2(T^2)$ with $\int_{T^2} p(s)\mathrm{d}s \neq 0$ and $\delta \in \mathbb{R}$ with $\delta \neq 0$ and $|\delta|$ sufficiently small, there exists a solution on the local centre manifold given by a heteroclinic connection between equilibria.
By proposition 3.1, since $\sigma\left(\mathcal{A}|_{X_c}\right) = \{0\}$, we can find solutions on the local centre manifold by finding solutions to the equation

$$\begin{aligned} U^c_\tau &= \pi_c F(U^c + \psi(U^c, \delta), \delta) \\ \delta_\tau &= 0; \end{aligned}$$

such that $\left(U^c(\tau) + \psi\left(U^c(\tau), \delta(\tau)\right), \delta(\tau)\right) \in \Omega$ for all $\tau \in \mathbb{R}$.
In order to find bounded solutions to this equation we will rewrite the first equation as a real scalar ordinary differential equation by letting $U^c(\tau) = y(\tau)U_0^+$, then we will use a blow-up rescaling to write the resulting equation as an ordinary differential equation with terms up to quadratic order plus a small perturbation. Then, since the constructed heteroclinic solutions persist under general small perturbation, we will find a solution for the original ordinary differential equation by finding a heteroclinic connection for the equation with only terms up to quadratic order.
To perform this rescaling and work out the equation with terms up to quadratic order, we first need to calculate the terms of the Taylor expansion, of the reduction map ψ, necessary to determine the equation on $X_c \times \mathbb{R}$ up to quadratic order in U^c.
The calculation of these terms requires a fairly long set of calculation which we present in appendix A. The result of these calculations is that we obtain the following expansion for the reduction map

$$\psi\left(U^c, \delta\right) = \left(L_0^-(\delta)U^c + Q_0^-\left(U^c, \delta\right) + O(\|U^c\|^3)\right)U_0^- + \sum_{m \in \mathbb{Z}^2 \setminus \{0\}} \psi_m^\pm\left(U^c, \delta\right)U_m^\pm, \tag{22}$$

where for $y \in \mathbb{R}$

$$\begin{aligned} L_0^-(\delta)\left(yU_0^+\right) &= a_0^- y \text{ with } a_0^- = \frac{(c^2 - 2\delta) - \sqrt{(c^2 - 2\delta) - 4\delta^2}}{2\delta}, \\ Q_0^-\left(yU_0^+, \delta\right) &= b_0^- y^2 \text{ with } b_0^- = \frac{q_2 p_0\left(1 + a_0^-\right)^3}{c^2 - 3\delta\left(1 + a_0^-\right)} \end{aligned}$$

and

$$\psi_m^\pm\left(U^c, \delta\right) = O(\|U^c\|^2) \text{ for } m \in \mathbb{Z}^2 \setminus \{0\}.$$

Now that we have this expansion we are in a position to write down the equation on X_c up to quadratic order and use this equation to find solutions on the local centre manifold. The ordinary differential equation on $X_c \times \mathbb{R}$ is

$$\begin{aligned} U^c_\tau &= \pi_c F(U^c + \psi(U^c,\delta),\delta) \\ \delta_\tau &= 0. \end{aligned}$$

Thus using the expansions of $\pi_c F$ in terms of the eigenvectors $\left\{U^\pm_m : m \in \mathbb{Z}^2\right\}$ (which can be seen in equation (54) in appendix A) and the expansion of the reduction map ψ given above, we can write this equation as

$$\begin{aligned} U^c_\tau = &-\frac{\delta}{\lambda^+_0 - \lambda^-_0}\left(u^c_1 + L^-_0(\delta)U^c + Q^-_0\left(U^c,\delta\right)\right)U^+_0 \\ &\qquad - \frac{q_2 p_0}{\lambda^+_0 - \lambda^-_0}\left(u^c_1 + L^-_0 U^c\right)^2 U^+_0 + O\left(\|U^c\|^3\right) \\ \delta_\tau = &\ 0. \end{aligned}$$

Now if we let $U^c = yU^+_0$ then, since $\lambda_0 + = 0$, $\lambda^-_0 = -c$, $L^-_0(\delta)\left(yU^-_0\right) = a^-_0 y$ and $Q^-_0\left(yU^c,\delta\right) = b^-_0 y^2$, we can write the equation on X_c in terms of y as

$$\begin{aligned} y_\tau &= -\frac{\delta}{c}\left(1 + a^-_0\right)y - \left(\frac{\delta b^-_0 + q_2 p_0\left(1+a^-_0\right)^2}{c}\right)y^2 + O\left(y^3\right) \\ \delta_\tau &= 0. \end{aligned} \tag{23}$$

To find a solution of the above ordinary differential equation we perform a blow-up rescaling using the small parameter δ. Thus if we let

$$\tilde{y} = \frac{y}{\delta} \text{ and } \tilde{\tau} = \delta\tau$$

then

$$\begin{aligned} \tilde{y}_{\tilde{\tau}} &= -\frac{\tilde{y} + q_2 p_0 \tilde{y}^2}{c} - \delta\left(\frac{a^-_0\tilde{y} + q_2 p_0 a^-_0\left(a^-_0 + 2\right)\tilde{y}^2}{\delta c} + \frac{b^-_0 \tilde{y}^2}{c} + O\left(\tilde{y}^3\right)\right) \\ \delta_{\tilde{\tau}} &= 0. \end{aligned} \tag{24}$$

and, since $a^-_0 = O(\delta)$, we can find solutions to the above equation by viewing it as a perturbation of the equation

$$\begin{aligned} \tilde{y}_{\tilde{\tau}} &= -\frac{\tilde{y} + q_2 p_0 \tilde{y}^2}{c} \\ \delta_{\tilde{\tau}} &= 0, \end{aligned} \tag{25}$$

for $|\delta|$ small.

Thus if we can find solutions for (25) which persist under general small perturbations, then there will be a corresponding solution for the perturbed equation provided $|\delta|$ is sufficiently small.

Now $p_0 = \frac{1}{|T^2|}\int_{T^2} p(s)\mathrm{d}s \neq 0$ so equation (25) has two equilibria $\tilde{y} = 0$ and $\tilde{y} = -1/q_2 p_0$, with 0 stable and $-1/q_2 p_0$ unstable. Hence there exists a heteroclinic connection between these two equilibria given by

$$\tilde{y}_1(\tau) = \begin{cases} \frac{-e^{-\frac{\tau}{c}}}{1 + q_2 p_0 \mathrm{e}^{-\frac{\tau}{c}}} & \text{if } p_0 q_2 > 0, \\ \frac{e^{-\frac{\tau}{c}}}{1 - q_2 p_0 \mathrm{e}^{-\frac{\tau}{c}}} & \text{if } p_0 q_2 < 0. \end{cases}$$

Thus, as heteroclinic connections between stable and unstable hyperbolic equilibria persist under small pertubations, there will be a heteroclinic connection between two equilibria for the perturbed equation (24) provided δ is sufficiently small.

If we denote the heteroclinic connection between the equilibria of the perturbed equation by $\tilde{y}_2(\tilde{\tau})$ then

$$\tilde{y}_2(\tilde{\tau}) \to \tilde{y}^\pm \text{ as } \tilde{\tau} \to \pm\infty.$$

Now inverting the blow-up rescaling we get a solution of the original equation (23)

$$y(\tau) = \delta\tilde{y}_2\left(\delta\tau\right)$$

such that
$$y(\tau) \to \delta\tilde{y}^{\pm} \text{ as } \tau \to \pm\infty$$
for a fixed $\delta \in \mathbb{R}$ with $|\delta| > 0$ sufficiently small. This then corresponds to a solution on $X_c \times \mathbb{R}$ given by
$$(U^c(\tau), \delta(\tau)) = \left(y(\tau)U_0^+, \delta\right).$$
Finally to show that $(U^c + \psi(U^c, \delta), \delta)$ is a solution on the local centre manifold we need to show that $(U^c(\tau) + \psi(U^c(\tau), \delta), \delta) \in \Omega$, the open neighbourhood of the origin from proposition 3.1, for all $\tau \in \mathbb{R}$.
Since $y(\tau)$ scales with δ it follows that
$$\sup_{\tau\in\mathbb{R}} \|U^c(\tau)\|_X \to 0 \text{ as } \delta \to 0.$$
Hence, as $\psi(0,0) = 0$ and ψ is continuous, it follows that for $|\delta| > 0$ sufficiently small $(U^c(\tau) + \psi(U^c(\tau), \delta), \delta) \in \Omega$ for all $\tau \in \mathbb{R}$. So for $\delta \in \mathbb{R}$ with $|\delta|$ sufficiently small there exists a heteroclinic connection which is mapped on to the local centre manifold.
Thus by proposition 3.1 for δ sufficiently small we have a solution for the spatial dynamical system, which correspond to a heteroclinic connection between equilibria. Hence we have proved the first part of theorem 2.1.

4.2. **Solutions on the Two Dimensional Centre Manifold for $\mu = \frac{1}{\varepsilon^2}\eta^T A\eta + \delta$.** Now we move on to look at the case when $\chi(\eta) = 2$. For this case $\mu = \frac{1}{\varepsilon^2}\eta^T A\eta + \delta$ (for $\eta \in \mathbb{Z}^2$) and the equation $m^T Am = \eta^T A\eta$ has only two solutions in $\mathbb{Z}^2$, given by $m = \pm\eta \in \mathbb{Z}^2$. Therefore
$$X_c = \mathrm{Span}_{\mathbb{C}}\left\{U_\eta^+, U_{-\eta}^+\right\} \cap Z = \left\{\alpha U_\eta^+ + \overline{\alpha}U_{-\eta}^+ : \alpha \in \mathbb{C}\right\}.$$
Our aim will be to show that there exists an open subset of periodic functions $p \in H^2(T^2)$ for which there is a non-trivial global solution on the local centre manifold. After this we will show that there is an open set contained within this open set for which we have the a solution corresponding to a heteroclinic connection between equilibria.
Again by proposition 3.1, since $\sigma(\mathcal{A}|_{X_c}) = \{0\}$, we can find global solutions on the local centre manifold by finding solutions to the equation
$$\begin{aligned} U_\tau^c &= \pi_c F(U^c + \psi(U^c, \delta), \delta) \\ \delta_\tau &= 0; \end{aligned}$$
such that $(U^c(\tau) + \psi(U^c(\tau), \delta(\tau)), \delta(\tau)) \in \Omega$ for all $\tau \in \mathbb{R}$.
To find solutions for this equation we will rewrite the first equation as a scalar complex ordinary differential equation by letting $U^c(\tau) = z(\tau)U_\eta^+ + \overline{z}(\tau)U_{-\eta}^+$. Once we have done this we will then use a blow-up rescaling using the small parameter δ to write this equation as a ordinary differential equation with terms up to quadratic order plus a small perturbation. Then we will rewrite the complex ordinary differential equation with terms up to quadratic order as a two dimensional real ordinary differential equation and use Conley index to find a non-trivial bounded solution that will persist under perturbations. It then follows that such a solution will correspond to a solution for the original equation for $\delta \in \mathbb{R}$ with $|\delta| > 0$ sufficiently small.
As in the previous case before we can perform this argument, we need to work out the terms of the Taylor expansion of the reduction map necessary to find the equation on X_c up to quadratic order in U^c. As for the previous case the calculation of these terms requires a long calculation which can be found in appendix B. The result of these calculation is that we obtain the following expansion of the reduction map
$$\begin{aligned} \psi(U^c, \delta) = &\left(L_\eta^-(\delta)U^c + Q_\eta^-(U^c, \delta) + O(\|U^c\|^3)\right)U_\eta^- \\ &+ \left(L_{-\eta}^-(\delta)U^c + Q_{-\eta}^-(U^c, \delta) + O(\|U^c\|^3)\right)U_{-\eta}^- \\ &+ \sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}} \psi_m^{\pm}(U^c, \delta)\, U_m^{\pm}, \end{aligned} \tag{26}$$

where if $U^c = zU_\eta^+ + \overline{z}U_{-\eta}^+$

$$
\begin{aligned}
L_\eta^-(\delta)\left(zU_\eta^+ + \overline{z}U_{-\eta}^+\right) &= \alpha_\eta^+ z, \\
L_{-\eta}^-(\delta)\left(zU_\eta^+ + \overline{z}U_{-\eta}^+\right) &= \overline{L_\eta^-(\delta)\left(zU_\eta^+ + \overline{z}U_{-\eta}^+\right)}, \\
Q_\eta^-\left(L_\eta^-(\delta)\left(zU_\eta^+ + \overline{z}U_{-\eta}^+\right), \delta\right) &= \gamma_\eta^- z^2 + 2\zeta_\eta^- |z|^2 + \sigma_\eta^- \overline{z}^2 \\
Q_{-\eta}^-\left(L_\eta^-(\delta)\left(zU_\eta^+ + \overline{z}U_{-\eta}^+\right), \delta\right) &= \overline{Q_\eta^-\left(L_\eta(\delta)\left(zU_\eta^+ + \overline{z}U_{-\eta}^+\right), \delta\right)}
\end{aligned}
$$

and

$$
\psi_m^\pm(U^c, \delta) = O(\|U^c\|^2) \text{ for } m \neq \pm\eta,
$$

for

$$
\alpha_\eta^- = \frac{-\left(2\delta + \lambda_\eta^-\left(\lambda_\eta^+ - \lambda_\eta^-\right)\right) + \sqrt{\left(2\delta + \lambda_\eta^-\left(\lambda_\eta^+ - \lambda_\eta^-\right)\right)^2 - 4\delta^2}}{2\delta}
$$

and γ_η^-, ζ_η^- and σ_η^- satisfying the equations

$$
\begin{aligned}
&\left(\lambda_\eta^-\left(\lambda_\eta^+ - \lambda_\eta^-\right) + 3\delta\left(1 + \alpha_\eta^-\right)\right)\gamma_\eta^- = -q_2 p_{-\eta}\left(1 + \alpha_\eta^-\right)^3, \\
&\left(\lambda_\eta^-\left(\lambda_\eta^+ - \lambda_\eta^-\right) + 2\delta\left(1 + \alpha_\eta^-\right) + \delta\frac{\left(1 + \overline{\alpha_\eta^-}\right)\left(\lambda_\eta^+ - \lambda_\eta^-\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right)\zeta_\eta^- \\
&\qquad = -q_2 p_\eta\left(1 + \alpha_\eta^-\right)^2\left(1 + \overline{\alpha_\eta^-}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\lambda_\eta^-\left(\lambda_\eta^+ - \lambda_\eta^-\right) + \delta\left(1 + \alpha_\eta^-\right) + 2\delta\frac{\left(1 + \overline{\alpha_\eta^-}\right)\left(\lambda_\eta^+ - \lambda_\eta^-\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right)\sigma_\eta^- \\
&\qquad = -q_2 p_{3\eta}\left(1 + \alpha_\eta^-\right)\left(1 + \overline{\alpha_\eta^-}\right)^2.
\end{aligned}
$$

With this expansion we are now in a position to write down the equation on X_c up to quadratic order in U^c and then use a blow-up rescaling to find the equation we need to analyse.
The equation on $X_c \times \mathbb{R}$ is

$$
\begin{aligned}
&U_\tau^c = \pi_c F\left(U^c + \psi\left(U^c, \delta\right), \delta\right) \\
&\delta_\tau = 0.
\end{aligned}
$$

Therefore, if we let $U^c(\tau) = z(\tau)U_\eta^+ + \overline{z}(\tau)U_{-\eta}^+$ and use the equation for the projection of F onto X_c (which is given in equation (58) in appendix B) together with expressions for the linear and quadratic terms from the last section, this equation becomes

$$\begin{aligned}
&z_\tau U_\eta^+ + \overline{z}_\tau U_{-\eta}^+ \\
&\quad = -\delta\left(\frac{z+\alpha_\eta^- z+\gamma_\eta^- z^2+2\zeta_\eta^-\left|z\right|^2+\sigma_\eta^-\overline{z}^2}{\lambda_\eta^+-\lambda_\eta^-}\right)U_\eta^+ \\
&\quad - \delta\left(\frac{\overline{z}+\overline{\alpha_\eta^-}\,\overline{z}+\overline{\sigma_\eta^-}z^2+2\overline{\zeta_\eta^-}\left|z\right|^2+\overline{\gamma_\eta^-}\,\overline{z}^2}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)U_{-\eta}^+ \\
&\quad - q_2\left(\frac{p_{-\eta}\left(1+\alpha_\eta^-\right)^2 z^2+2p_\eta\left|1+\alpha_\eta^-\right|^2\left|z\right|^2+p_{3\eta}\left(1+\overline{\alpha_\eta^-}\right)^2\overline{z}^2}{\lambda_\eta^+-\lambda_\eta^-}\right)U_\eta^+ \\
&\quad - q_2\left(\frac{p_{-3\eta}\left(1+\alpha_\eta^-\right)^2 z^2+2p_{-\eta}\left|1+\alpha_\eta^-\right|^2\left|z\right|^2+p_\eta\left(1+\overline{\alpha_\eta^-}\right)^2\overline{z}^2}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)U_{-\eta}^+ \\
&\quad + O\left(z^3\right) \\
&\delta_\tau = 0.
\end{aligned}$$

Thus, as the equation on the $U_{-\eta}^+$ component is just the complex conjugate of the equation on the U_η^+ component, we can solve the equation on $X_c \times \mathbb{R}$ by studying the system of equations

$$\begin{aligned}
z_\tau = &-\frac{\delta\left(1+\alpha_\eta^-\right)z+\left(q_2p_{-\eta}\left(1+\alpha_\eta^-\right)^2+\delta\gamma_\eta^-\right)z^2+2\left(q_2p_\eta\left|1+\alpha_\eta^-\right|^2+\delta\zeta_\eta^-\right)\left|z\right|^2}{\lambda_\eta^+-\lambda_\eta^-} \\
&-\frac{\left(q_2p_{3\eta}\left(1+\overline{\alpha_\eta^-}\right)^2+\delta\sigma_\eta^-\right)\overline{z}^2}{\lambda_\eta^+-\lambda_\eta^-}+O\left(z^3\right)
\end{aligned}$$

$$\delta_\tau = 0. \tag{27}$$

The next step is to perform a blow-up rescaling using the small parameter δ to work out the equation that we will study. Thus if we let

$$\tilde{z} = \frac{z}{\delta} \text{ and } \tilde{\tau} = \delta\tau;$$

then the above equation becomes

$$\begin{aligned}
\tilde{z}_{\tilde{\tau}} = &-\frac{\tilde{z}+q_2\left(p_{-\eta}\tilde{z}^2+2p_\eta\left|\tilde{z}\right|^2+p_{3\eta}\overline{\tilde{z}}^2\right)}{\lambda_\eta^+-\lambda_\eta^-} \\
&-\delta\Bigg(\frac{\alpha_\eta^-\tilde{z}+q_2p_{-\eta}\left(\left(1+\alpha_\eta^-\right)^2-1\right)\tilde{z}^2+2q_2p_\eta\left(\left|1+\alpha_\eta^-\right|^2-1\right)\left|\tilde{z}\right|^2}{\delta\left(\lambda_\eta^+-\lambda_\eta^-\right)} \\
&+\frac{q_2p_{3\eta}\left(\left(1+\overline{\alpha_\eta^-}\right)^2-1\right)\overline{\tilde{z}}^2}{\delta\left(\lambda_\eta^+-\lambda_\eta^-\right)}+\frac{\gamma_\eta^-\tilde{z}^2+2\zeta_\eta^-\left|z\right|^2+\sigma_\eta^-\overline{\tilde{z}}^2}{\lambda_\eta^+-\lambda_\eta^-}+O\left(\tilde{z}^3\right)\Bigg)
\end{aligned}$$

$$\delta_{\tilde{\tau}} = 0. \tag{28}$$

Now, since $\alpha_\eta^- = O\left(\delta\right)$, we can find solutions to the above equation by viewing it as a perturbation of the equation

$$\begin{aligned}
\tilde{z}_{\tilde{\tau}} &= -\frac{\tilde{z}+q_2\left(p_{-\eta}\tilde{z}^2+2p_\eta\left|\tilde{z}\right|^2+p_{3\eta}\overline{\tilde{z}}^2\right)}{\lambda_\eta^+-\lambda_\eta^-} \\
\delta_{\tilde{\tau}} &= 0.
\end{aligned} \tag{29}$$

Thus if we can find a solution for (29) which persists under small perturbations, then there will be a corresponding solution for the perturbed equation (28) provided δ is sufficiently small.

The final step to obtain the required equation is to write the complex ordinary differential equation (29) as a system of real ordinary differential equations. Hence if we let $\tilde{z}(\tilde{\tau}) = y_1(\tilde{\tau}) + \mathrm{i}y_2(\tilde{\tau})$ then (29) becomes,

$$\frac{dy_1}{d\tilde{\tau}} + \mathrm{i}\frac{dy_2}{d\tilde{\tau}} = -\frac{y_1 + \mathrm{i}y_2}{\lambda_\eta^+ - \lambda_\eta^-} - q_2\left(\frac{p_{-\eta}(y_1+\mathrm{i}y_2)^2 + 2p_\eta\left(y_1^2+y_2^2\right) + p_{3\eta}(y_1-\mathrm{i}y_2)^2}{\lambda_\eta^+ - \lambda_\eta^-}\right)$$
$$\delta_{\tilde{\tau}} = 0.$$

Now taking real and complex parts we turn the first equation into two real equation to get

$$\begin{aligned}
\frac{dy_1}{d\tilde{\tau}} =& -\mathrm{Re}\left\{\frac{1}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1 + \mathrm{Im}\left\{\frac{1}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_2 - q_2\mathrm{Re}\left\{\frac{p_{-\eta}+2p_\eta+p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1^2 \\
&+ 2q_2\mathrm{Im}\left\{\frac{p_{-\eta}-p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1y_2 - q_2\mathrm{Re}\left\{\frac{2p_\eta - p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_2^2 \\
\frac{dy_2}{d\tilde{\tau}} =& -\mathrm{Im}\left\{\frac{1}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1 - \mathrm{Re}\left\{\frac{1}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_2 - q_2\mathrm{Im}\left\{\frac{2p_\eta - p_{-\eta}+p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1^2 \\
&- 2q_2\mathrm{Re}\left\{\frac{p_{-\eta}-p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1y_2 - q_2\mathrm{Im}\left\{\frac{2p_\eta - p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_2^2 \\
\delta_{\tilde{\tau}} =& 0.
\end{aligned} \tag{30}$$

Thus we can find solutions to the equation on $X_c \times \mathbb{R}$ by finding solutions to the above equation which persist under perturbations. In the next subsection we will study equations of this type using Conley Index.

4.3. **Constructing Solutions via Conley Index.** In this subsection we will consider the question of the existence of bounded solutions for ordinary differential equations of the type derived in the previous section. For convenience we will write these ordinary differential equation in the following ways

$$\begin{aligned}
\frac{dy_1}{d\tau} &= \lambda_1 y_1 - \lambda_2 y_2 + \mathbf{y}^T M\mathbf{y} =: f_1(\mathbf{y}) \\
\frac{dy_2}{d\tau} &= \lambda_2 y_1 + \lambda_1 y_2 + \mathbf{y}^T N\mathbf{y} =: f_2(\mathbf{y}),
\end{aligned} \tag{31}$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ N = \begin{pmatrix} n_1 & n_2 \\ n_2 & n_3 \end{pmatrix} \text{ and } M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix}.$$

Remark 4.2.
- We ignore the equation $\delta_\tau = 0$ for the moment, since this equation just tells us that δ is constant for solutions on the centre manifold.
- The coefficient λ_1 and λ_2 will correspond to the values given in equation (30), but the matrices M and N can be any symmetric matrices, since any possible matrices can be generated by choosing an appropriate periodic function p.

We will use Conley index to prove the existence of bounded solutions for this ordinary differential equation which persist under perturbations, see [7, 22]. Our strategy for proving the existence of non-trivial bounded solutions will be the following: First we work out the Conley index of an isolating block which contains the equilibria of the equation. Once we have this Conley index we compare it to what the Conley index would be if there where no non-trivial bounded solutions. Thus if these two indices are different, then there must be a non-trivial bounded solution. However this method only works when we are able to work out the Conley index. In this paper we give two sets of conditions on the nonlinearity for which we can calculate the Conley index, for work on a case where we are not always able to calculate the Conley index see [5]. However the cases in the present paper give existence for an open set of periodic function $p \in H^2(T^2)$.

The Conley index is calculated as follows: Take a large rectangle which contains the equilibria of the ordinary differential equation. Now look at the sections of the boundary through which the vector field points in (entry set) and out (exit set). We then try to connect the entry sets to the exit set using solution curves. More precisely we replace the rectangle with a set B whose boundary consists of non-trivial segments of entry points and exit points, and non-trivial segments of solutions curves that connect an entry point to an exit point. If we can do this we have constructed an isolating block and we can calculate the Conley index by quotienting by the exit set and taking the homotopy equivalence class, for more details see [7, 22]. We illustrate this process in the example below.

Example 4.3. Consider the ordinary differential equation

$$\frac{dy_1}{d\tau} = y_1 - y_2 + y_1^2 + y_2^2$$
$$\frac{dy_2}{d\tau} = y_1 + y_2 + y_1^2 - y_2^2.$$

This equation has four equilibria $(0,0)$, $(0,1)$, $(0,-1)$ and $(-1,1)$. We want to consider a rectangle which contains these equilibria, so let

$$R = \{(y_1, y_2) : -3 \leqslant y_1 \leqslant 2, -3 \leqslant y_2 \leqslant 4\}$$

and look at the direction of the vector field round the edge of this rectangle see figure 1.

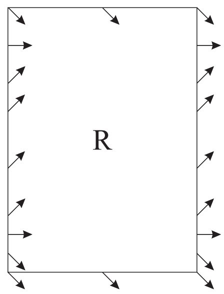


Figure 1. *Direction of vector field around edge of the rectangle R.*

Now to make this rectangle into a isolating block B we need to connect the entry sets to the exit sets using solution curves, the result of this process can be seen in figure 2.

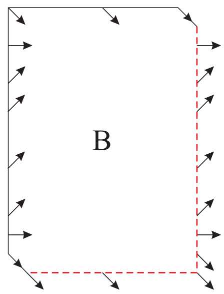


Figure 2. *The isolating block B constructed from the rectangle R, with the exit set b^+ marked by the red dashed line.*

The isolating block shown in figure 2 has an exit set which consists of only one subsection of the boundary, thus when we define the Conley index of the isolated invariant set S inside B as $h(S) = [B, b^+]$ meaning the homotopy type of the pointed space obtained by collapsing the subset b^+ of B to a point. As we can retract B with b^+ collapsed to a one-point pointed space, which is denoted by $\overline{0}$, we obtain

$$h(S) = \left[B, b^+\right] = \overline{0}.$$

The above example illustrates how we can workout the Conley index for a particular ordinary differential equation. It is a example of the following series of steps which we will use to workout the Conley index for our general ordinary differential equation.

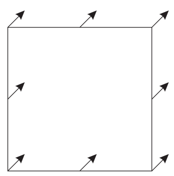
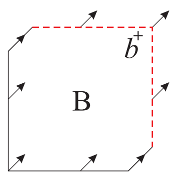
(1) Take a large rectangle which contains the equilibria of the equation, then consider different cases depending on how many times the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ intersect the boundary of the rectangle.
(2) For each of these cases work out the possible arrangements of the points where the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ intersect the boundary. Choosing the shape of the rectangles to reduce the number of cases as much as possible.
(3) Now for each arrangement draw all the possible vector fields around the edge of the rectangle and turn these into isolating blocks when this is possible.
(4) Workout the Conley index from these isolating blocks.

Following the above steps we consider the problem in two cases, when the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ intersect the boundary of a sufficiently large rectangle, 0 and 4, times and $\det M, \det N \neq 0$. We deal with each of these cases in separate lemmata.

Remark 4.4. The sign of $\det M$ and $\det N$ determine what kind of conic section the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ will be. If the determinant is greater than zero then the curve will be a circle or ellipse, on the other hand if the determinant is less than zero then the curve will be a hyperbola.

Lemma 4.5. *Let* $\det N > 0$ *and* $\det M > 0$ *then the Conley index of an isolating block which contains the equilibria of equation* (31) *will be equal to* $[([b^+], [b^+])] =: \overline{0}$.

Proof. Since $\det N > 0$ and $\det M > 0$ the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ correspond to circles or ellipses. Thus it is possible to choose a sufficiently large rectangle, such that these curves will not intersect the boundary of the rectangle. Thus we get a generic vector field shown in the table below, as f_1 and f_2 do not change sign on the boundary of the rectangle. Then we can construct an isolating block and work out the Conley index.

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
		$[B, b^+] = \overline{0}$

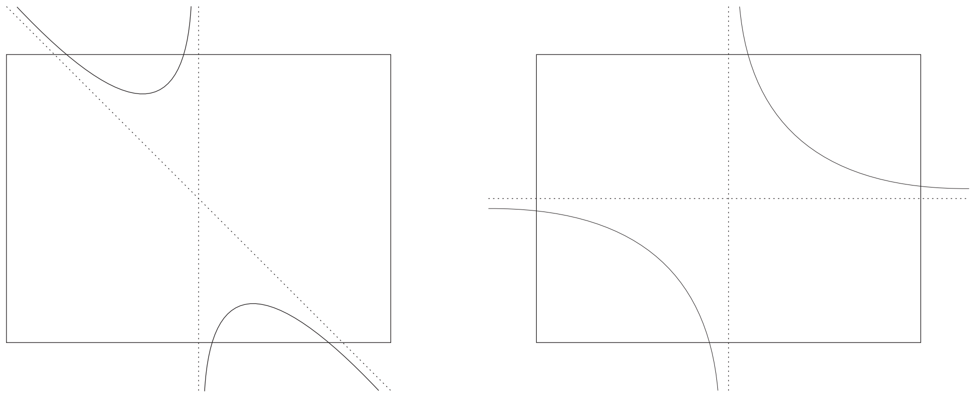
Here and through out this section the exit set b^+ is marked by the red dashed line.
Any other phase plane will be a rotation of this one and thus have the same Conley index. □

Next we turn our attention to the case when the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ intersect the boundary of the rectangle 4 times.

Lemma 4.6. *Let* $\det M \cdot \det N < 0$*, then the Conley index of an isolating block containing the equilibria of equation* (31) *will be equal to* $[([b^+], [b^+])] = \overline{0}$.

Proof. Without loss of generality let $\det N < 0$ and $\det M > 0$, if this is not the case just interchange the roles of y_1 and y_2.
In this case the curve $f_1(\mathbf{y}) = 0$ corresponds to an ellipse or circle and again we can choose a rectangle large enough that this curve will not intersect its boundary. However the curve $f_2(\mathbf{y}) = 0$ will correspond to a hyperbola which will intersect the boundary at 4 points for a sufficiently large rectangle.
Thus we can split this case into two situations. As $|\mathbf{y}|$ gets large the hyperbola $f_2(\mathbf{y}) = 0$ will approach its asymptotes, if neither of these asymptotes are parallel to the y_1 axis then we can always choose a rectangle such that the curve $f_2(\mathbf{y}) = 0$ intersects the top and bottom of the rectangle twice, we will call this situation 1 and it is illustrated in figure 3(a). On the other hand if one of the asymptotes is parallel to the y_1 axis then it is always possible to choose a

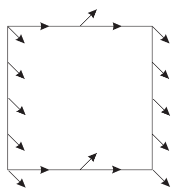
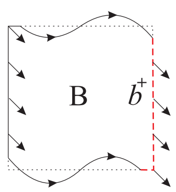
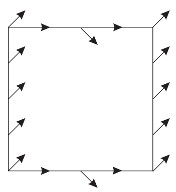
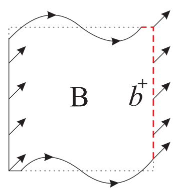
rectangle such that the curve $f_2(\mathbf{y}) = 0$ intersects each side once, we will call this situation 2 and it is illustrated in figure 3(b).



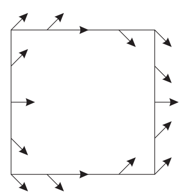
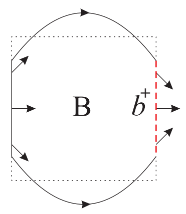
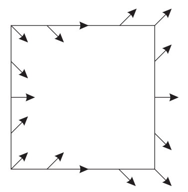
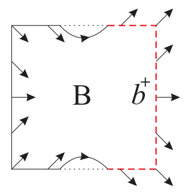
(a) Situation 1 (b) Situation 2

Figure 3. *Situations for lemma 4.6.*

Now for situation 1 the vector field around the edge of the rectangle, up to reflection, will look like one of the cases given in the table below.

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block
		$[B, b^+] = \overline{0}$
		$[B, b^+] = \overline{0}$

Again here and through out this section the exit set b^+ is marked by the red dashed line.
The other two possibilities with the vector field pointing to the left are just reflections of these two cases.
On the other hand for situation 2 the vector field, up to reflection will look like one of the cases given in the table below.

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block
		$[B, b^+] = \overline{0}$
		$[B, b^+] = \overline{0}$

Thus we have shown that for all vector field satisfying the conditions of the lemma the Conley index is $\overline{0}$. ☐

The case $\det M < 0$ and $\det N < 0$ involves a large number of subcases, for some of which the Conley index was computed in [5]. Now, that we have calculated some possible Conley indices, we move onto the main point of this section: proving the existence of non-trivial solutions.

Proposition 4.7. *If the Conley index of an invariant set containing the equilibria of the ordinary differential equation* (31) *is* $\overline{0}$ *then the ordinary differential equation has a non-trivial bounded solution connecting to* 0.

Proof. We prove this statement via a contradiction argument. Suppose that the equilibrium 0 is an isolated invariant set. Then we can split the invariant set into two disjoint isolated invariant sets 0 and I. So following Smoller [22, Theorem 22.31] we have that the Conley index, which we understand as the homotopy equivalence class of a pointed space, of the (disjoint) union of the two isolated invariant sets is the wedge or sum of the two pointed spaces

$$h\left(\{0\} \sqcup I\right) = h(0) \vee h(I).$$

However we know that the equilibrium 0 is either a repeller or sink by direct inspection and thus

$$h(0) = \Sigma^0 \text{ or } \Sigma^2,$$

where Σ^k is the pointed k-sphere. Now $\Sigma^0 \vee h(I)$ and $\Sigma^2 \vee h(I)$ are not homotopic to a point, so the Conley index of an isolating block containing 0 and I is not equal to what the Conley index would be if 0 was isolated, which is a contradiction. Thus there must be a non-trivial solution which connects to zero, either as $\tau \to \infty$ or $\tau \to -\infty$. ☐

Thus we have proved the existence of a non-trivial bounded solution for the ordinary differential equation (31), however we do not know what type of solution we have found.
The Poincaré-Bendixson theorem implies that this solution must be a periodic, heteroclinic or homoclinic orbit. In the next result we give conditions on the quadratic terms which ensure that the solution we have found will be a heteroclinic connection between two equilibria. For the convenience of the reader we recall an equivalent form of equation (31),

$$\begin{aligned}
\frac{dy_1}{d\tau} &= \lambda_1 y_1 - \lambda_2 y_2 + m_1 y_1^2 + 2m_2 y_1 y_2 + m_3 y_2^2 \\
\frac{dy_2}{d\tau} &= \lambda_2 y_1 + \lambda_1 y_2 + n_1 y_1^2 + 2n_2 y_1 y_2 + n_3 y_2^2.
\end{aligned}$$

Proposition 4.8. *Suppose that equation* (31) *has a non-trivial bounded solution then, this solution will be a heteroclinic connection between two equilibria if the following conditions hold*

$$\lambda_1 m_2 + \lambda_2 n_2 = \lambda_1 n_1 - \lambda_2 m_1$$
$$\lambda_1 n_2 - \lambda_2 m_2 = \lambda_1 m_3 + \lambda_2 n_3$$

Proof. We prove this result by showing that under the above assumptions the ordinary differential equation (31) has a gradient structure i.e.

$$\mathbf{y}_\tau = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \nabla K(\mathbf{y})$$

for some function $K : \mathbb{R}^2 \to \mathbb{R}$. If this is the case then K will be an increasing or decreasing function along solutions. This then eliminates the possibility of periodic and homoclinic orbits, so by the Poincaré-Bendixson theorem the only thing the non-trivial solution can be is a heteroclinic connection between two equilibria. Thus all we need to do is construct K.
Now, for equation (31), we have

$$\begin{aligned}\mathbf{y}_\tau &= \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} m_1 y_1^2 + 2m_2 y_1 y_2 + m_3 y_2^2 \\ n_1 y_1^2 + 2n_2 y_1 y_2 + n_3 y_2^2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \Bigg(\mathbf{y} + \\ &\quad \frac{1}{\lambda_1^2 + \lambda_2^2} \begin{pmatrix} (\lambda_1 m_1 + \lambda_2 n_1) y_1^2 + 2(\lambda_1 m_2 + \lambda_2 n_2) y_1 y_2 + (\lambda_1 m_3 + \lambda_2 n_3) y_2^2 \\ (\lambda_1 n_1 - \lambda_2 m_1) y_1^2 + 2(\lambda_1 n_2 - \lambda_2 m_2) y_1 y_2 + (\lambda_1 n_3 - \lambda_2 m_3) y_2^2 \end{pmatrix} \Bigg).\end{aligned}$$

Thus, using the assumption from the statement of the proposition, we can just take

$$\begin{aligned}K(\mathbf{y}) =& \frac{y_1^2 + y_2^2}{2} + \frac{1}{\lambda_1^2 + \lambda_2^2} \Bigg(\frac{(\lambda_1 m_1 + \lambda_2 n_1)}{3} y_1^3 + (\lambda_1 m_2 + \lambda_2 n_2) y_1^2 y_2 \\ &+ (\lambda_1 m_3 + \lambda_2 n_3) y_1 y_2^2 + \frac{(\lambda_1 n_3 - \lambda_2 m_3)}{3} y_2^3 \Bigg).\end{aligned}$$

Hence, under the assumptions of the proposition the equation has a gradient structure and the non-trivial solution must be a heteroclinic connection between two equilibria. ☐

Remark 4.9. Furthermore, heteroclinic connections between hyperbolic equilibria persist under small perturbations if at least one is a sink or repeller. Thus there will exist heteroclinic connections connecting to 0 for equations which are close to equations satisfying the conditions of the preceding lemma. Hence we have an open set around the choices of nonlinearity which satisfy the conditions of the above lemma for which the solution we find on the centre manifold is a heteroclinic connection between equilibria.

4.4. **Solutions on the Centre Manifold.** In the previous section we showed that for the two-dimensional real version of the unperturbed equation (30), we have the existence of a non-trivial bounded solutions. This solution will correspond to a solution of the unperturbed complex ordinary differential equation (29), as the Conley index arguments are stable under small δ perturbations.
Then for $\delta \in \mathbb{R}$ with $|\delta| > 0$ sufficiently small there will be a corresponding solution for the perturbed complex ordinary differential equation (28). Let us denote this solution by $\tilde{z}(\tilde{\tau})$. Now if we reverse the blow-up rescaling we get a solution to the original complex ordinary differential equation (27)

$$z(\tau) = \delta \tilde{z}(\delta \tau),$$

for $\delta \in \mathbb{R}$ with $|\delta| > 0$ sufficiently small. This solution then gives a solution to the equation on $X_c \times \mathbb{R}$

$$(U^c(\tau), \delta(\tau)) = \left(\delta \tilde{z}(\delta\tau) U_\eta^+ + \delta \overline{\tilde{z}}(\delta\tau) U_{-\eta}^+, \delta \right),$$

for $|\delta| > 0$ sufficiently small.

Thus, to find a solution on the centre manifold, we just need to ensure that $(U^c(\tau) + \psi(U^c(\tau), \delta), \delta) \in \Omega$ for all $\tau \in \mathbb{R}$.
Now, since $\tilde{z}$ is a bounded solution, we have that

$$\sup_{\tau \in \mathbb{R}} \|U^c(\tau)\|_X \to 0 \text{ as } \delta \to 0.$$

Thus, as $\psi(0,0) = 0$ and ψ is continuous, it follows that for δ sufficiently small $(U^c(\tau) + \psi(U^c(\tau), \delta), \delta) \in \Omega$ for all $\tau \in \mathbb{R}$. Hence for sufficiently small δ there exist a non-trivial solution which is mapped on to the local centre manifold. Furthermore if the conditions of proposition 4.8 are satisfied then the solution will be a heteroclinic connection between possibly spatial periodic equilibria. Hence, by proposition 3.1, for δ sufficiently small we have a non-trivial solution for the spatial dynamical system and we have completed the proof of the second part of theorem 2.1.

□

5. Homogenisation via Centre Manifold

In this section we will prove theorem 2.3 which deals with the homogenisation of the generalised travelling wave solutions we have found for a particular case. The proof of this result will be based on the method we used to construct generalised travelling wave solutions in the proof of theorem 2.1. The key idea will be to formulate the problem as a spatial dynamical system in such away that we can control how the reduction map for the local centre manifold changes as $\varepsilon \to 0$. Once this is done we will obtain our desired result by showing that the dynamics on the centre manifold converge as $\varepsilon \to 0$.

5.1. **Rescaled Generalised Travelling Wave Solutions.** Thus we begin by formulating the problem in such a way that we can remove the singular dependence on ε while controlling how the spectral gap either side of the imaginary axis changes as $\varepsilon \to 0$. In order to do this we look at rescaled generalised travelling waves solutions

$$u(x,y) = v^\varepsilon\left(x \cdot k - ct, \frac{x}{\varepsilon}\right) = w^\varepsilon\left(\frac{x \cdot k - ct}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

where $w^\varepsilon = w^\varepsilon(\tilde{\tau}, \xi)$ is periodic in ξ with periodic cell $[0, 2\pi]^2$.
The idea will be to prove the existence of rescaled generalised travelling wave solutions and determine what happens to them as $\varepsilon \to 0$. This will then tell us what happens to the generalised travelling wave solutions as $\varepsilon \to 0$.
Hence if we substitute the rescaled generalised travelling wave ansatz into the reaction diffusion equation (1) we get an equation in terms of the profile function w^ε

$$0 = \frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi\left(A\nabla_\xi w^\varepsilon\right) + 2k^T A \nabla_\xi w^\varepsilon_{\tilde{\tau}} + w^\varepsilon_{\tilde{\tau}\tilde{\tau}}\right) + \frac{c}{\varepsilon} w^\varepsilon_{\tilde{\tau}} + \delta w^\varepsilon + p(\xi)\, q(w^\varepsilon),$$

which can be rearranged to get

$$0 = \mathrm{div}_\xi\left(A\nabla_\xi w^\varepsilon\right) + 2k^T A \nabla_\xi w^\varepsilon_{\tilde{\tau}} + w^\varepsilon_{\tilde{\tau}\tilde{\tau}} + c\varepsilon w^\varepsilon_{\tilde{\tau}} + \varepsilon^2\left(\delta w^\varepsilon + p(\xi)\, q(w^\varepsilon)\right). \tag{32}$$

Notice that in the second of these equations we have removed all the singular dependence on ε.

5.2. **Spatial Dynamical System Formulation and Centre Manifold Reduction.** Now to prove the existence of rescaled generalised travelling wave solutions we formulate this equation as a spatial dynamical system treating $\tilde{\tau}$ as the time variable and δ as a extra dependent variable. Thus letting $W^\varepsilon = (w^\varepsilon, w^\varepsilon_{\tilde{\tau}})$ gives us the spatial dynamical system

$$\begin{aligned} W^\varepsilon_{\tilde{\tau}} &= \mathcal{J}_\varepsilon W^\varepsilon + \varepsilon^2 F\left(W^\varepsilon, \delta\right) \\ \delta_{\tilde{\tau}} &= 0, \end{aligned} \tag{33}$$

where

$$\mathcal{J}_\varepsilon = \begin{pmatrix} 0 & 1 \\ -\mathrm{div}_\xi\left(A\nabla_\xi \cdot\right) & -c\varepsilon - 2k^T A \nabla_\xi \end{pmatrix}$$

and

$$F\left(W^{\varepsilon},\delta\right)=\left(\begin{array}{c}0\\ -\delta w^{\varepsilon}-pq(w^{\varepsilon})\end{array}\right).$$

Notice that this spatial dynamical system (33) is the same as the spatial dynamical system (7) in subsection 3.1 if we choose the wave speed c to be $c\varepsilon$ in the travelling wave ansatz (4), set ε to equal to 1 in the linear part of (7) and multiplying the nonlinearity by ε^2.
Thus via a similar calculation to the one done in section 3 we see that the linear operator $\mathcal{J}_\varepsilon$ has eigenvalues

$$\hat{\lambda}^{\pm}_{m,\varepsilon}=\frac{-\left(c\varepsilon+2\mathrm{i}k^TAm\right)\pm\sqrt{\left(c\varepsilon+2\mathrm{i}k^TAm\right)^2+4m^TAm}}{2}, \tag{34}$$

with associated eigenfunctions

$$W^{\pm}_m=\left(\begin{array}{c}1\\ \hat{\lambda}^{\pm}_{m,\varepsilon}\end{array}\right)\exp\left(\mathrm{i}m\cdot\xi\right).$$

Therefore if we define the spaces

$$Z_\varepsilon:=\left\{\sum_{m\in\mathbb{Z}^2}\alpha^{\pm}_mW^{\pm}_m:\overline{\alpha^{\pm}_m}=\alpha^{\pm}_{-m},\sum_{m\in\mathbb{Z}^2}\left|\alpha^{\pm}_m\right|^2\left(1+\left|\hat{\lambda}^{\pm}_{m,\varepsilon}\right|^4\right)<\infty\right\}$$

and

$$X_\varepsilon:=\left\{\sum_{m\in\mathbb{Z}^2}\alpha^{\pm}_mW^{\pm}_m:\overline{\alpha^{\pm}_m}=\alpha^{\pm}_{-m},\sum_{m\in\mathbb{Z}^2}\left|\alpha^{\pm}_m\right|^2\left(1+\left|\hat{\lambda}^{\pm}_{m,\varepsilon}\right|^6\right)<\infty\right\},$$

it follows from proposition 3.1 that we have a centre manifold reduction for each fixed $\varepsilon>0$. Thus the following result holds:

Proposition 5.1. *For $\varepsilon>0$ fixed there exist a finite dimensional subspace $X^{\varepsilon}_c\times\mathbb{R}\subset X_\varepsilon\times\mathbb{R}$ and a projection π_c onto X^{ε}_c. Letting $X^{\varepsilon}_h=(Id-\pi_c)\left(X_\varepsilon\right)$ there exists a neighbourhood of the origin $\Omega_\varepsilon\subset X_\varepsilon\times\mathbb{R}$ and a map $\psi^{\varepsilon}\in C^2_b\left(X^{\varepsilon}_c\times\mathbb{R},X^{\varepsilon}_h\right)$ with $\psi^{\varepsilon}(0,0)=0$ and $D\psi^{\varepsilon}(0,0)=0$, such that if $(W^{c,\varepsilon},\delta):I\to X_c\times\mathbb{R}$ solves*

$$\begin{aligned}W^{c,\varepsilon}_\tau&=\mathcal{J}_\varepsilon W^{c,\varepsilon}+\varepsilon^2\pi_cF\left(W^{c,\varepsilon}+\psi^{\varepsilon}\left(W^{c,\varepsilon},\delta\right),\delta\right),\\ \delta_\tau&=0\end{aligned}$$

for some interval $I\subset\mathbb{R}$, and $(W^{\varepsilon},\delta)(\tau)=(W^{c,\varepsilon}(\tau)+\psi^{\varepsilon}(W^{c,\varepsilon}(\tau),\delta(\tau)),\delta(\tau))\in\Omega_\varepsilon$ for all $\tau\in I$ then (W^{ε},δ) solves

$$\begin{aligned}W^{\varepsilon}_\tau&=\mathcal{J}_\varepsilon W^{\varepsilon}+\varepsilon^2F(W^{\varepsilon},\delta)\\ \delta_\tau&=0.\end{aligned}$$

Furthermore, from lemmata 3.8 and 3.12, we have that $\mathcal{J}_\varepsilon\in\mathcal{L}\left(X_\varepsilon,Z_\varepsilon\right)$, the nonlinearity $F\in C^2\left(X_\varepsilon\times\mathbb{R},X_\varepsilon\right)$ and $\sigma\left(\mathcal{J}_\varepsilon\right)=\left\{\hat{\lambda}^{\pm}_m:m\in\mathbb{Z}^2\right\}$.
Thus, from the equation for the eigenvalues (34), it follows that the only eigenvalues with zero real part are $\hat{\lambda}^{+}_{0,\varepsilon}=0$ if $c>0$ and $\hat{\lambda}^{-}_{0,\varepsilon}=0$ if $c<0$, so we have that

$$X^{\varepsilon}_c=\begin{cases}\mathrm{Span}_{\mathbb{R}}\left\{W^{+}_0\right\} & \text{if } c>0\\ \mathrm{Span}_{\mathbb{R}}\left\{W^{-}_0\right\} & \text{if } c<0\end{cases}.$$

Also if we define the sets and spans of eigenfunctions

$$\mathcal{S}^s=\left\{U^{\pm}_m|\mathrm{Re}\hat{\lambda}^{\pm}_{m,\varepsilon}<0\right\},\qquad S^s=\mathrm{span}_{\mathbb{C}}\left\{\mathcal{S}^s\right\}\cap Z_\varepsilon,$$

$$\mathcal{S}^u=\left\{U^{\pm}_m|\mathrm{Re}\hat{\lambda}^{\pm}_{m,\varepsilon}>0\right\},\qquad S^u=\mathrm{span}_{\mathbb{C}}\left\{\mathcal{S}^u\right\}\cap Z_\varepsilon,$$

then we can define subspaces X_s^ε, X_u^ε and X_h^ε of X and Z_s^ε, Z_u^ε and Z_h^ε of Z as the closures of S^s, S^u and $S^h := S^s \cup S^u$ in X_ε and Z_ε respectively. Furthermore projections π_s, π_u and $\pi_h \in \mathcal{L}(Z_\varepsilon)$ and $\mathcal{L}(X_\varepsilon)$ on to the relevant spaces can be constructed in a similar way to how π_c was constructed in section 3.

Now proposition 5.1 tells us that for each fixed $\varepsilon > 0$ there exists a reduction map ψ^ε and an open neighbourhood of the origin Ω_ε for which the centre manifold property holds. Therefore the next step is to understand what happens to the reduction map and the open neighbourhood of the origin as $\varepsilon \to 0$.

We answer this question by showing that there exists a fixed open neighbourhood of the origin $\Omega \subset X_\varepsilon \times \mathbb{R}$, which together with ψ^ε has the centre manifold property, and that the reduction map ψ^ε converges to zero uniformly as $\varepsilon \to 0$.

To verify these statements we will carefully construct the reduction map and show that it has the properties we want.

5.3. **Construction of Reduction Map.** The first step towards these goals is to understand how the spectral gap either side of the imaginary axis depends on ε. Therefore we need to find a lower bound on the absolute values of the real parts of the non-zero eigenvalues.

Lemma 5.2. *There exists a constant $\hat{C} > 0$ such that*
$$\left|Re\lambda_{m,\varepsilon}^{\pm}\right| \geqslant \hat{C}\varepsilon,$$
for all $Re\lambda_{m,\varepsilon}^{\pm} \neq 0$.

Proof. If $m = 0$ then either $\hat{\lambda}_0^- = -c\varepsilon$ or $\hat{\lambda}_0^+ = -c\varepsilon$ depending on the sign of c and thus the absolute value is bounded below by $|c|\,\varepsilon$. On the other hand if $m \neq 0$ then we need to estimate the absolute value of
$$\mathrm{Re}\hat{\lambda}_{m,\varepsilon}^{\pm} = \frac{-c\varepsilon \pm \mathrm{Re}\sqrt{(c\varepsilon + 2\mathrm{i}k^TAm)^2 + 4m^TAm}}{2}.$$
We do this by first estimating the absolute value of the real part of the square root. In order to do this we observe that, since $k \in S_A^1$, there exists a vector $k_\perp \in S_A^1$ such that the set $\{k, k_\perp\}$ forms an orthonormal basis for $\mathbb{R}^2$ with respect to the inner product $(x,y)_A = x^TAy$. So using this basis we can write $m = (m,k)_A k + (m,k_\perp)_A k_\perp$ for all $m \in \mathbb{Z}^2$ and then it follows that
$$m^TAm = (m,k)_A^2 + (m,k_\perp)_A^2.$$
Therefore have that for all $m \neq 0$ and $\varepsilon \in (0,1)$
$$\begin{aligned}
&\left|\mathrm{Re}\sqrt{(c\varepsilon + 2\mathrm{i}k^TAm)^2 + 4m^TAm}\right| \\
&= \left|\mathrm{Re}\sqrt{c^2\varepsilon^2 + 4(m,k_\perp)_A^2 + 4\mathrm{i}c\varepsilon(m,k)_A}\right| \\
&= \frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(c^2\varepsilon^2 + 4(m,k_\perp)_A^2\right)^2 + \left(4c\varepsilon(m,k)_A\right)^2} + c^2\varepsilon^2 + 4(m,k_\perp)_A^2} \\
&\geqslant \frac{1}{\sqrt{2}}\sqrt{\sqrt{c^4\varepsilon^4 + 8c^2\varepsilon^2 m^TAm} + c^2\varepsilon^2} \\
&\geqslant \left(\sqrt{\frac{c^2 + \sqrt{c^4 + 8c^2m^TAm}}{2}}\right)\varepsilon \qquad (\text{as } 0 < \varepsilon < 1).
\end{aligned}$$
Now, as A is a symmetric positive definite matrix, there exists an $m_\star \in \mathbb{Z}^2$ such that $m_\star^TAm_\star = \min_{m\in\mathbb{Z}^2\setminus\{0\}} m^TAm > 0$, and thus it follows from the above estimate that
$$\left|\mathrm{Re}\sqrt{(c\varepsilon + 2\mathrm{i}k^TAm)^2 + 4m^TAm}\right| \geqslant \left(\sqrt{\frac{c^2 + \sqrt{c^4 + 8c^2m_\star^TAm_\star}}{2}}\right)\varepsilon,$$

for all $m \in \mathbb{Z}^2 \setminus \{0\}$. From this estimate and the fact that the only other non-zero eigenvalue is equal to $-c\varepsilon$ we get that

$$\left|\mathrm{Re}\hat{\lambda}^{\pm}_{m,\varepsilon}\right| \geqslant \underbrace{\min\left\{\left(\sqrt{\frac{c^2+\sqrt{c^4+8c^2m_\star^TAm_\star}}{2}}-|c|\right),|c|\right\}}_{=:\hat{C}}\varepsilon,$$

for all $\mathrm{Re}\hat{\lambda}^{\pm}_{m,\varepsilon} \neq 0$. □

Thus we see that the size of the spectral gaps either side of the imaginary axis go to zero with order ε as $\varepsilon \to 0$. Hence we have estimated how the spectral gap depends on ε, we can now use this information to determine how the solution operator for the affine problem on X_h^ε, which was described in hypothesis (H3) in section 3, depends on ε.

If we let $\gamma(\varepsilon) := \min\left\{\mathrm{Re}\hat{\lambda}^{\pm}_{m,\varepsilon} : \mathrm{Re}\hat{\lambda}^{\pm}_{m,\varepsilon} \neq 0\right\}$, then the affine problem is the following; for each $\eta \in [0, \gamma(\varepsilon))$ find a map $K_h^\varepsilon \in \mathcal{L}\left(C_\eta(\mathbb{R}, X_h^\varepsilon)\right)$ such that if $W^{h,\varepsilon} = K_h^\varepsilon f$, then $W^{h,\varepsilon}$ is the unique solution in $C_\eta(\mathbb{R}, X_h^\varepsilon)$ of

$$W^{h,\varepsilon}_{\tilde{\tau}} = \mathcal{J}_\varepsilon W^{h,\varepsilon} + f.$$

From lemma 3.19 we have that there exist exponentially decaying C_0-semigroups $\hat{T}_s(\tilde{\tau})$ and $\hat{T}_u(\tilde{\tau})$ on Z_s^ε and Z_u^ε which decay with rate $\gamma(\varepsilon)$. Therefore from the proof of lemma 3.21 we have that K_h^ε exists

$$K_h^\varepsilon f(\tilde{\tau}) = \int_{-\infty}^{\tilde{\tau}} \hat{T}_s(\tilde{\tau}-\sigma)\pi_s f(\sigma)\mathrm{d}\sigma - \int_{\tilde{\tau}}^{\infty} T_u(\sigma-\tilde{\tau})\pi_u f(\sigma)\mathrm{d}\sigma,$$

and we have the estimate

$$\|K_h^\varepsilon\|_{\mathcal{L}\left(C_\eta(\mathbb{R},X_h^\varepsilon)\right)} \leqslant \frac{2}{\gamma(\varepsilon)-\eta}.$$

With this information we are now in a position to start constructing the reduction map for the local centre manifold. In the following calculations we will give a brief outline of how the reduction map is constructed in order to determine how it depends on ε. More detail of the construction can be found in the proof of [24, Theorem 1].

The first step in constructing the reduction map for the local centre manifold is to apply a smooth cut-off function to the nonlinearity to get the equation

$$\begin{aligned} &W^\varepsilon_{\tilde{\tau}} = \mathcal{J}_\varepsilon W^\varepsilon + \varepsilon^2\chi\left(W^\varepsilon,\delta\right)F(W^\varepsilon,\delta) =: \mathcal{J}_\varepsilon W^\varepsilon + \varepsilon^2 G(W^\varepsilon,\delta) \\ &\delta_{\tilde{\tau}} = 0, \end{aligned} \tag{35}$$

where χ is a smooth cut-off function which is independent of ε, such that G is a C^1-bounded and globally Lipschitz function, and $\chi(W^\varepsilon,\delta) = 1$ for $(W^\varepsilon,\delta) \in \Omega \subset X_\varepsilon \times \mathbb{R}$ an open neighbourhood of the origin (details of how to construct this cut-off function can be found in the proof of [24, Theorem 3]).

Now that we have equation (35) the main idea is to formulate this equation as an abstract equation on a suitable Banach space. We are then able to use Banach's contraction mapping principle to find solutions to this abstract problem which are then used to construct the reduction map.

If we let $\zeta = \gamma(\varepsilon)/2$ then we can formulate (35) as the following abstract problem; for $\delta \in \mathbb{R}$ fixed find $W^\varepsilon \in C_\zeta(\mathbb{R}, X_\varepsilon)$ such that

$$W^\varepsilon = \pi_c W^\varepsilon(0) + \varepsilon^2 K^\varepsilon G(W^\varepsilon,\delta),$$

where $K^\varepsilon \in \mathcal{L}\left(C_\zeta(\mathbb{R}, X_\varepsilon)\right)$ is defined by

$$K^\varepsilon V(\tilde{\tau}) = \int_0^{\tilde{\tau}} \pi_c V(s)\mathrm{d}s + K_h^\varepsilon\left(\pi_h V\right)(\tilde{\tau})$$

for which we have the estimate,

$$\|K^\varepsilon\|_{\mathcal{L}\left(C_\zeta(\mathbb{R},X_\varepsilon)\right)} \leqslant \frac{1}{\zeta} + \frac{2}{\gamma(\varepsilon)-\zeta} \leqslant \frac{6}{\gamma(\varepsilon)} \leqslant \frac{6}{\hat{C}\varepsilon}.$$

Now if we consider the equation

$$W^\varepsilon = W^0 + \varepsilon^2 K^\varepsilon G(W^\varepsilon, \delta),$$

for $W^0 \in C_\zeta(\mathbb{R}, X_\varepsilon)$, then for $\varepsilon > 0$ sufficiently small we have that

$$\left|\varepsilon^2 K^\varepsilon G\right|_{\text{lip}} \leqslant \frac{6\varepsilon}{\hat{C}} |G|_{\text{lip}} < 1,$$

and thus it follows from Banach's contraction mapping theorem that there exists a map $\Psi^\varepsilon : C_\zeta(\mathbb{R}, X_\varepsilon) \times \mathbb{R} \to C_\zeta(\mathbb{R}, X_\varepsilon)$ such that,

$$\Psi^\varepsilon(W,\delta) = W^0 + \varepsilon^2 K^\varepsilon G\left(\Psi^\varepsilon(W,\delta), \delta\right). \tag{36}$$

Then using this map Ψ^ε we have constructed, we define the reduction map to be

$$\psi^\varepsilon(W^c,\delta) := \pi_h \Psi^\varepsilon(W^c,\delta)(0) = \varepsilon^2 K_h^\varepsilon\left(\pi_h G\left(\Psi^\varepsilon(W^c,\delta),\delta\right)\right)(0). \tag{37}$$

For which we have the estimate

$$\begin{aligned} \|\psi^\varepsilon(W^c,\delta)\|_X &\leqslant \varepsilon^2 \|K^\varepsilon\|_{\mathcal{L}\left(C_\zeta(\mathbb{R},X_\varepsilon)\right)} \|\pi_h\|_{\mathcal{L}(X_\varepsilon)} \|G\|_{C(X_\varepsilon,X_\varepsilon)} \\ &\leqslant \frac{6\varepsilon}{\hat{C}} \|G\|_{C(X_\varepsilon,X_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{38}$$

Now it is shown in the proof of [24, Theorem 1] that the reduction map constructed in this way is in $C_b^{0,1}(X_c^\varepsilon \times \mathbb{R}, X_h^\varepsilon)$. Then, since $G \in C^2(X_\varepsilon \times \mathbb{R}, X_\varepsilon)$ and the Lipschitz constant of $\varepsilon^2 G$ converges to zero faster than the spectral gaps either side of the imaginary axis as $\varepsilon \to 0$, it follows from [24, Theorem 2] that $\psi^\varepsilon \in C_b^2(X_c^\varepsilon \times \mathbb{R}, X_h^\varepsilon)$ for sufficiently small $\varepsilon > 0$.
We are also able to determine how the derivative with respect to W^c of the reduction map ψ^ε behaves as $\varepsilon \to 0$. To do this we first need to find the derivative of Ψ^ε with respect to W. This is found by differentiating equation (36) with respect to W to get

$$D_W \Psi^\varepsilon(W,\delta) = Id + \varepsilon^2 K^\varepsilon DG\left(\Psi^\varepsilon(W,\delta)\right) D_W \Psi^\varepsilon(W,\delta),$$

which, for sufficiently small $\varepsilon > 0$, we can rearrange to obtain

$$\begin{aligned} D_W \Psi^\varepsilon(W,\delta) &= \left(Id - \varepsilon^2 K^\varepsilon DG\left(\Psi^\varepsilon(W,\delta)\right)\right)^{-1} \\ &= \sum_{n=0}^{\infty} \varepsilon^{2n} \left(K^\varepsilon DG\left(\Psi^\varepsilon(W,\delta)\right)\right)^n. \end{aligned}$$

Therefore if we differentiate equation (37) with respect to W^c we see that,

$$D_{W^c}\psi^\varepsilon(W^c,\delta) = \varepsilon^2 K_h^\varepsilon \pi_h D_W G\left(\Psi^\varepsilon(W^c,\delta),\delta\right) D\Psi^\varepsilon(W^c,\delta),$$

which will tend to 0 uniformly as $\varepsilon \to 0$ by a similar estimate to (38).
Thus we have shown that for all sufficiently small $\varepsilon > 0$ there exist a local centre manifold reduction with a fixed open neighbourhood of the origin $\Omega \subset X_\varepsilon \times \mathbb{R}$ and a reduction map $\psi^\varepsilon(W^c,\delta) = O(\varepsilon)$ as $\varepsilon \to 0$.

5.4. **Convergence of Dynamics on the Centre Manifold.** Now that we have determined the behaviour of the reduction map as $\varepsilon \to 0$, we move on to study what happens to the rescaled generalised travelling wave solutions as $\varepsilon \to 0$. To achieve this we investigate the convergence of the dynamics on the centre manifold as $\varepsilon \to 0$. Thus we will firstly show that for all $\varepsilon > 0$ sufficiently small there exists a rescaled travelling wave solution which corresponds to a heteroclinic connection between equilibria. After this we will show that these rescaled generalised travelling wave solutions converge to a travelling wave solution which does not depend on the spatial variable as $\varepsilon \to 0$

Thus we start by finding solutions $(W^{c,\varepsilon},\delta)$ of the ordinary differential equation on $X_c^\varepsilon\times\mathbb{R}$

$$\begin{aligned} W^{c,\varepsilon}_{\tilde{\tau}} &= \varepsilon^2\pi_c F\left(W^{c,\varepsilon}+\psi^\varepsilon\left(W^{c,\varepsilon},\delta\right),\delta\right)\\ \delta_{\tilde{\tau}} &= 0, \end{aligned} \tag{39}$$

such that $(W^{c,\varepsilon}(\tilde{\tau})+\psi^\varepsilon\left(W^{c,\varepsilon}(\tilde{\tau}),\delta\right),\delta)\in\Omega$ for all $\tilde{\tau}\in\mathbb{R}$. Now as the second of these equations just tells us that $\delta\in\mathbb{R}$ is a fixed constant, we can treat δ as a fixed constant and just deal with the first equation.
Thus as

$$X_c^\varepsilon = \begin{cases}\mathrm{Span}_{\mathbb{R}}\left\{W_0^+\right\} & \text{if } c>0\\ \mathrm{Span}_{\mathbb{R}}\left\{W_0^-\right\} & \text{if } c<0\end{cases},$$

we will assume without loss of generality that $c>0$ and $X_c=\mathrm{Span}_{\mathbb{R}}\left\{W_0^+\right\}$ throughout this section, if $c<0$ then the calculation from this section will still give the same result with the roles of W_0^+ and W_0^- interchanged.
Thus if we let $W^c=yW_0^+$ and

$$\psi^\varepsilon\left(yW_0^+,\delta\right) =: \begin{pmatrix}\phi_1^\varepsilon(y,\delta)\\ \phi_2^\varepsilon(y,\delta)\end{pmatrix},$$

then the first equation of equation (39) becomes the ordinary differential equation

$$\begin{aligned} y_{\tilde{\tau}}W_0^+ &= \varepsilon^2\pi_c\begin{pmatrix}0\\ -\delta\left(y+\phi_1^\varepsilon(y,\delta)\right)-pq\left(y+\phi_1^\varepsilon(y,\delta)\right)\end{pmatrix}\\ &= \varepsilon\left(\frac{-\delta y-p_0q(y)}{c}\right)W_0^+ + \varepsilon^2\pi_c\begin{pmatrix}0\\ -\delta\phi_1^\varepsilon(y,\delta)-p\left(q\left(y+\phi_1^\varepsilon(y,\delta)\right)-q(y)\right)\end{pmatrix}. \end{aligned}$$

and if we let $\theta=\varepsilon\tilde{\tau}$ then this equation becomes

$$y_\theta W_0^+ = \left(\frac{-\delta y-p_0q(y)}{c}\right)W_0^+ + \varepsilon\pi_c\begin{pmatrix}0\\ -\delta\phi_1^\varepsilon(y,\delta)-p\left(q\left(y+\phi_1^\varepsilon(y,\delta)\right)-q(y)\right)\end{pmatrix}. \tag{40}$$

Now we want to show that for $\delta\in\mathbb{R}$ with $|\delta|>0$ sufficiently small fixed and $\varepsilon>0$ sufficiently small this equation has a heteroclinic connection between two equilibria. We will do this by using a perturbation argument, thus we start by considering the equation

$$y_\theta = -\frac{\delta y+p_0q(y)}{c}. \tag{41}$$

If we use the rescaling

$$\tilde{y}=\frac{y}{\delta} \text{ and } \tilde{\theta}=\delta\theta,$$

then this equation becomes

$$\tilde{y}_{\tilde{\theta}} = -\frac{\tilde{y}+q_2p_0\tilde{y}^2+\delta O\left(\tilde{y}^3\right)}{c}, \tag{42}$$

which for $|\delta|$ small can be viewed as a perturbation of the equation

$$\tilde{y}_{\tilde{\theta}} = -\frac{\tilde{y}+q_2p_0\tilde{y}^2}{c}. \tag{43}$$

This equation has a heteroclinic orbit between the hyperbolically stable equilibrium $\tilde{y}=0$ and hyperbolically unstable equilibrium $\tilde{y}=-1/q_2p_0$ given by

$$\tilde{y}_1(\tilde{\theta}) = \begin{cases}\frac{-e^{-\frac{\tilde{\theta}}{c}}}{1+q_2p_0\mathrm{e}^{-\frac{\tilde{\theta}}{c}}} & \text{if } p_0q_2>0,\\ \frac{e^{-\frac{\tilde{\theta}}{c}}}{1-q_2p_0\mathrm{e}^{-\frac{\tilde{\theta}}{c}}} & \text{if } p_0q_2<0.\end{cases}$$

Thus, since heteroclinic connections between unstable and stable hyperbolic equilibria persist under small perturbations, it follows that for $|\delta|$ sufficiently small there will be a heteroclinic connection between two equilibria for the perturbed system (42). If we denote this heteroclinic by $\tilde{y}_2(\tilde{\theta})$ then

$$\tilde{y}_2(\tilde{\theta}) \to \tilde{y}_2^{\pm} \text{ as } \tilde{\theta} \to \pm\infty,$$

and if we undo the rescaling we performed we get a heteroclinic connection for equation (41) given by, $y^0(\theta) = \delta\tilde{y}_2(\delta\theta)$.
This heteroclinic connection between unstable and stable hyperbolic equilibria is a solution for the ordinary differential equation

$$y^0_\theta W_0^+ = \left(\frac{-\delta y^0 - p_0 q(y^0)}{c}\right) W_0^+, \tag{44}$$

which is part of equation (40). Therefore to prove the existence of a heteroclinic connection for equation (40) all we need to show is that the remaining parts of equation (40) are a small C^1-perturbation of equation (44) for $\varepsilon > 0$ sufficiently small, since if this is the case then the heteroclinic will persist for small values of $\varepsilon > 0$.
Thus we need to show that

$$P^\varepsilon(y) := \varepsilon\pi_c \begin{pmatrix} 0 \\ -\delta\phi_1^\varepsilon(y,\delta) - p\left(q\left(y + \phi_1^\varepsilon(y,\delta)\right) - q(y)\right) \end{pmatrix}$$

and its derivative tend to zero as $\varepsilon \to 0$ in a neighbourhood of our heteroclinic connection y^0.
If we choose $\varepsilon_0 > 0$ then there exist intervals I and J such that

$$\left\{y^0(\theta) : \theta \in \mathbb{R}\right\} \subset\subset I$$

and

$$I \cup \{y + \phi_1^\varepsilon(y,\delta) : y \in I, \varepsilon \in (0,\varepsilon_0)\} \subset\subset J.$$

Then since q is a differentiable function it will be a Lipschitz function on the interval J with Lipschitz constant L_q. So for $y \in I$ we have that

$$|q(y + \phi_1^\varepsilon(y,\delta)) - q(y)| \leqslant L_q\,|\phi_1^\varepsilon(y,\delta)| = O(\varepsilon) \text{ as } \varepsilon \to 0,$$

from estimate (38). Thus it follows from this estimate together with estimate (38) that

$$P^\varepsilon(y) = \varepsilon\pi_c \begin{pmatrix} 0 \\ -\delta\phi_1^\varepsilon(y,\delta) - p\left(q\left(y + \phi_1^\varepsilon(y,\delta)\right) - q(y)\right) \end{pmatrix} = O(\varepsilon) \text{ as } \varepsilon \to 0,$$

uniformly on I.
Now if we differentiate P^ε with respect to y we get

$$\begin{aligned} D_y P^\varepsilon(y) =& \varepsilon\pi_c \begin{pmatrix} 0 \\ -\delta D_y\phi_1^\varepsilon(y,\delta) \end{pmatrix} \\ &+ \varepsilon\pi_c \begin{pmatrix} 0 \\ -p\left(q'\left(y + \phi_1^\varepsilon(y,\delta)\right) - q'(y) + q'\left(y + \phi_1^\varepsilon(y,\delta)\right) D_y\phi_1^\varepsilon(y,\delta)\right) \end{pmatrix} \end{aligned}$$

and thus since q is a differentiable function we have that q' is a Lipschitz function on J, so as $D_y\phi_1^\varepsilon(y,\delta) \to 0$ uniformly as $\varepsilon \to 0$, it follows by a similar argument to the one used for P^ε, that $D_y P^\varepsilon(y) \to 0$ as $\varepsilon \to 0$ on I.
Thus, since the heteroclinic $y^0(\theta)$ connects an unstable and stable hyperbolic equilibrium for equation (44) and P^ε is a small C^1-perturbation for sufficiently small $\varepsilon > 0$, there will exist a heteroclinic connection $y^\varepsilon(\theta)$ between equilibria which solve equation (40). Furthermore it follows from this perturbation argument that the equilibria which the heteroclinic $y^\varepsilon(\theta)$ connects will converge to those that the heteroclinic $y^0(\theta)$ connects as $\varepsilon \to 0$.
Now to confirm that this heteroclinic connection for the ordinary differential equation corresponds to a rescaled generalised travelling wave solution we need to check that it stays with in the open neighbourhood Ω when it is mapped onto the centre manifold. To do this we need information about the convergence of these heteroclinic solutions as $\varepsilon \to 0$. This information will also allow us to understand what happens to the rescaled generalised travelling wave solutions as $\varepsilon \to 0$.

Thus we want to look at the convergence of these heteroclinic connections y^ε as $\varepsilon \to 0$. For convenience we will define,

$$g(y) := -\frac{\delta y + p_0 q(y)}{c},$$

and

$$h^\varepsilon(y) W_0^+ := P^\varepsilon(y).$$

Then using this notation equation (40) becomes

$$y_\theta W_0^+ = g(y) W_0^+ + h^\varepsilon(y) W_0^+,$$

and so the ordinary differential equation we are interested in is

$$y_\theta = g(y) + h^\varepsilon(y).$$

From the perturbation argument we just performed we know that for $\delta \in \mathbb{R}$ with $|\delta|$ sufficiently small and $\varepsilon > 0$ sufficiently small there exists a heteroclinic connection between two equilibria $y^\varepsilon(\theta)$ such that

$$y_\theta^\varepsilon = g\left(y^\varepsilon\right) + h^\varepsilon\left(y^\varepsilon\right) \tag{45}$$

and

$$y^\varepsilon(\theta) \to y_\pm^\varepsilon \text{ as } \theta \to \pm\infty.$$

Also from our study of equation (41) in the perturbation argument we know that there exists a heteroclinic connection between equilibria $y^0(\theta)$ such that

$$y_\theta^0 = g\left(y^0\right) \tag{46}$$

and

$$y^0(\theta) \to y_\pm^0 \text{ as } \theta \to \pm\infty.$$

Finally the perturbation argument also tells us that the equilibria $y_\pm^\varepsilon \to y_\pm^0$ as $\varepsilon \to 0$.
Now with these preparations we state a convergence lemma.

Lemma 5.3. *For δ fixed,*

$$y^\varepsilon = y^0 + O(\varepsilon) \text{ as } \varepsilon \to 0,$$

holds uniformly on $\mathbb{R}$.

Proof. The proof of this result is based on the proof of [19, Theorem 5.3.1] with some minor adaptations to deal with the fact our heteroclinic solutions are defined on the whole of $\mathbb{R}$. The basic idea is to split $\mathbb{R}$ into a series of subintervals and then use a combination of the exponential attraction of solutions near hyperbolic equilibria and Gronwall type arguments to get the convergence we want on each subinterval.
Therefore we start by looking at the behaviour of solutions to (46) near the equilibria y_+^0 and y_-^0. Since y_+^0 and y_-^0 are hyperbolically stable and unstable equilibria respectively, by [19, Lemma 5.2.7] there exist constants $\rho > 0$, $M > 0$ and $\mu > 0$ such that if $\hat{y}^0$ and $\tilde{y}^0$ solve equation (46) then we have the following two properties:

(1) If $\hat{y}^0(\theta_0)$ and $\tilde{y}^0(\theta_0) \in B_\rho\left(y_+^0, \mathbb{R}\right)$ for $\theta_0 \in \mathbb{R}$ then

$$\left|\hat{y}^0(\theta) - \tilde{y}^0(\theta)\right| \leqslant M e^{-\mu|\theta-\theta_0|} \left|\hat{y}^0(\theta_0) - \tilde{y}^0(\theta_0)\right| \text{ for all } \theta \geqslant \theta_0. \tag{47}$$

(2) If $\hat{y}^0(\theta_0)$ and $\tilde{y}^0(\theta_0) \in B_\rho\left(y_-^0, \mathbb{R}\right)$ for $\theta_0 \in \mathbb{R}$ then

$$\left|\hat{y}^0(\theta) - \tilde{y}^0(\theta)\right| \leqslant M e^{-\mu|\theta-\theta_0|} \left|\hat{y}^0(\theta_0) - \tilde{y}^0(\theta_0)\right| \text{ for all } \theta \leqslant \theta_0. \tag{48}$$

Also, since $Me^{-\mu s} \to 0$ as $s \to \infty$, there will exist a $T_1 > 0$ such that

$$M \exp(-\mu s) \leqslant 1/2 \text{ for all } s \geqslant T_1. \tag{49}$$

Next we need estimates on when the heteroclinic connection y^0 and the equilibria $y^\varepsilon_\pm$ are close enough to $y^0_\pm$ that the exponential estimates (47) and (48) hold. Thus, as $y^0(\theta) \to y^0_\pm$ when $\theta \to \pm\infty$, there exists a $T_2 > 0$ such that

$$\left|y^0(\pm\theta) - y^0_\pm\right| < \frac{\rho}{2} \text{ for all } \theta \geqslant T_2, \tag{50}$$

and, since $y^\varepsilon_\pm \to y^0_\pm$ as $\varepsilon \to 0$ there exists a $\varepsilon_1 > 0$ such that

$$\left|y^\varepsilon_\pm - y^0_\pm\right| < \rho \text{ for all } \varepsilon \in (0, \varepsilon_1). \tag{51}$$

Now if we let $T = \max\{T_1, T_2\}$ then we split $\mathbb{R}$ into a union of finite intervals

$$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} I_n,$$

where

$$I_n = \begin{cases} [nT, (n+1)T] & \text{for } n > 0 \\ [-T, T] & \text{for } n = 0 \\ [(n-1)T, nT] & \text{for } n < 0 \end{cases}.$$

With this splitting of $\mathbb{R}$ into subintervals the first step towards proving the convergence we want is to use a Gronwall type argument to show that we have our desired convergence on the finite interval [-2T,2T]. This argument is based on the proof of [19, Lemma 1.5.3].

In order to show this result we first need to collect some properties of the functions g and h^ε. Due to the fact g is a differentiable function it follows that if we choose an interval $\hat{I}$ such that

$$\left\{y^0(\theta) : \theta \in \mathbb{R}\right\} \cup \{y^\varepsilon(\theta) : \theta \in \mathbb{R}, \varepsilon \in (0, \varepsilon_1)\} \subset\subset \hat{I},$$

then g will be a Lipschitz function on $\hat{I}$ with Lipschitz constant L_g. On the other hand using a similar argument to the one we used to show that $P^\varepsilon(y) = O(\varepsilon)$ on I as $\varepsilon \to 0$ we can show that $h^\varepsilon(y) = O(\varepsilon)$ on $\hat{I}$ as $\varepsilon \to 0$ and thus there will exist a constant $M_h > 0$ such that

$$|h^\varepsilon(y)| \leqslant M_h\varepsilon \text{ for all } y \in \hat{I}.$$

Also, since any translations of the heteroclinic connections y^0 and y^ε are also heteroclinic connections, if we choose

$$\tilde{y} \in \left\{y^0(\theta) : \theta \in [-2T, 2T]\right\} \cap \bigcap_{\varepsilon\in(0,\varepsilon_0)} \{y^\varepsilon(\theta) : \theta \in [-2T, T]\}$$

then by translating the heteroclinic connections y^0 and y^ε we can ensure that $y^0(0) = \tilde{y}$ and $y^\varepsilon(0) = \tilde{y}$.

Therefore for $\theta \in [-2T, 2T]$ and $\varepsilon \in (0, \varepsilon_0)$ we can use equations (45) and (46), and the fundamental theorem of calculus to write $y^\varepsilon(\theta)$ and $y^0(\theta)$ as

$$y^\varepsilon(\theta) = \tilde{y} + \int_0^\theta g\left(y^\varepsilon(\sigma)\right) + h^\varepsilon\left(y^\varepsilon(\sigma)\right) \mathrm{d}\sigma$$

and

$$y^0(\theta) = \tilde{y} + \int_0^\theta g\left(y^\varepsilon(\sigma)\right) \mathrm{d}\sigma.$$

Then from these representations it follows that

$$\begin{aligned}
\left|y^\varepsilon(\theta)-y^0(\theta)\right| &\leqslant \left|\int_0^\theta \left|g\left(y^\varepsilon(\sigma)\right)-g\left(y^0(\sigma)\right)\right| \mathrm{d}\sigma\right| + \left|\int_0^\theta \left|h^\varepsilon\left(y^\varepsilon(\sigma)\right)\right| \mathrm{d}\sigma\right| \\
&\leqslant L_g \left|\int_0^\theta \left|y^\varepsilon(\sigma)-y^0(\sigma)\right| \mathrm{d}\sigma\right| + |\theta|\, M_h \varepsilon \\
&\leqslant |\theta|\, M_h\varepsilon \exp\left(L_g |\theta|\right) \qquad \text{(by Gronwall's inequality)} \\
&\leqslant 2TM_h\varepsilon\exp\left(2L_gT\right) =: \Gamma(\varepsilon). \qquad (52)
\end{aligned}$$

Thus we have our desired convergence on the interval $[-2T, 2T]$. We now need to deal with the intervals further out.

We first need to find out when the heteroclinic y^ε is close enough to the equilibria $y^0_\pm$ for the exponential estimates (47) and (48) to hold. Hence, since $\Gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that

$$\Gamma(\varepsilon) \leqslant \frac{\rho}{2} \text{ for all } \varepsilon \in (0, \varepsilon_2).$$

Therefore if $\varepsilon \in (0, \varepsilon_2)$ then it follows from the estimates (50), (51) and (52) that

$$\left|y^\varepsilon(\theta) - y^0_+\right| < \rho \text{ for all } \theta \geqslant T$$

and

$$\left|y^\varepsilon(\theta) - y^0_-\right| < \rho \text{ for all } \theta \leqslant -T.$$

With this information we can now show the convergence we want on the rest of $\mathbb{R}$. We start by considering the interval I_n for $n \geqslant 2$. In order to get the estimates we want on this interval we need an intermediate solution, thus we define a solution $y^{0,n}$ of the ordinary differential equation (46) which has initial value $y^{0,n}((n-1)T) = y^\varepsilon((n-1)T)$. Then using a Gronwall type argument similar to the one done above we get that

$$\sup_{\theta \in I_n} \left|y^\varepsilon(\theta) - y^{0,n}(\theta)\right| \leqslant \Gamma(\varepsilon).$$

Thus it follows that

$$\begin{aligned}
\sup_{\theta \in I_n} \left|y^\varepsilon(\theta) - y^0(\theta)\right| &\leqslant \sup_{\theta \in I_n} \left|y^\varepsilon(\theta) - y^{0,n}(\theta)\right| + \sup_{\theta \in I_n} \left|y^{0,n}(\theta) - y^0(\theta)\right| \\
&\leqslant \Gamma(\varepsilon) + \sup_{\theta \in I_n} \left|y^{0,n}(\theta) - y^0(\theta)\right|,
\end{aligned}$$

and, since $\theta \geqslant nT$ for $\theta \in I_n$, $y^0((n-1)T) \in B_\rho\left(y^0_+, \mathbb{R}\right)$ and $y^{0,n}((n-1)T) = y^\varepsilon((n-1)T) \in B_\rho\left(y^0_+, \mathbb{R}\right)$, it follows from the exponential estimate (47) that

$$\begin{aligned}
\sup_{\theta \in I_n} \left|y^\varepsilon(\theta) - y^0(\theta)\right| &\leqslant \Gamma(\varepsilon) + \sup_{\theta \in I_n} Me^{-\mu|\theta-(n-1)T|} \left|y^\varepsilon((n-1)T) - y^0((n-1)T)\right| \\
&\leqslant \Gamma(\varepsilon) + \frac{1}{2} \sup_{\theta \in I_{n-1}} \left|y^\varepsilon(\theta) - y^0(\theta)\right|,
\end{aligned}$$

where the second inequality follows from inequality (49), as for $\theta \in I_n$ we have that $\theta - (n-1)T \geqslant T \geqslant T_2$. Now if we apply this inequality recursively we get that

$$\begin{aligned}
\sup_{\theta \in I_n} \left|y^\varepsilon(\theta) - y^0(\theta)\right| &\leqslant \left(1 + \frac{1}{2}\right)\Gamma(\varepsilon) + \left(\frac{1}{2}\right)^2 I_{(n-2)} \left|y^\varepsilon(\theta) - y^0(\theta)\right| \\
&\leqslant \left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^n\right)\Gamma(\varepsilon) \\
&\leqslant 2\Gamma(\varepsilon).
\end{aligned}$$

Notice that the last of these estimates is independent of ε and so will hold for all $n \geqslant 2$. Thus we have the convergence we want on the interval $[-2T, \infty)$.

A similar calculation to the one above will work for the interval $(-\infty, -2T]$. Hence we have that for sufficiently small $\varepsilon > 0$

$$\left|y^\varepsilon(\theta) - y^0(\theta)\right| \leqslant 2\Gamma(\varepsilon) \text{ for all } \theta \in \mathbb{R},$$

and

$$y^\varepsilon = y^0 + O(\varepsilon) \text{ as } \varepsilon \to 0,$$

which completes the proof of lemma 5.3. □

From this result we have that equation (40) has a heteroclinic connection between equilibria,

$$\hat{W}^{c,\varepsilon}(\theta) = y^\varepsilon(\theta)W_0^+ = y^0(\theta)W_0^+ + O(\varepsilon), \text{ as } \varepsilon \to 0.$$

Therefore, as $\theta = \varepsilon\tilde{\tau}$, it follows that the ordinary differential equation on the centre manifold (39) has a heteroclinic connection

$$W^{c,\varepsilon}(\tilde{\tau}) = y^\varepsilon(\varepsilon\tilde{\tau})W_0^+.$$

With this information we can now show that the heteroclinic connection we have found for the ordinary differential equation corresponds to a rescaled generalised travelling wave solution. We know from the perturbation argument earlier in this proof that $y^0(\theta) = \delta\tilde{y}_2(\delta\theta)$ and thus it will converge uniformly to 0 as $\delta \to 0$. Therefore, since $y^\varepsilon = y^0 + O(\varepsilon)$ as $\varepsilon \to 0$ and, ψ^ε is continuous with $\psi^\varepsilon(0, 0) = 0$, it follows that if $|\delta|$ and $\varepsilon > 0$ are sufficiently small then

$$(W^\varepsilon(\tilde{\tau}), \delta) = (W^{c,\varepsilon}(\tilde{\tau}) + \psi^\varepsilon\left(W^{c,\varepsilon}(\tilde{\tau}), \delta\right), \delta) \in \Omega \text{ for all } \tilde{\tau} \in \mathbb{R}.$$

Thus (W^ε, δ) solves the spatial dynamical system (33) for all $|\delta| > 0$ and $\varepsilon > 0$ sufficiently small. Hence, since $W^\varepsilon = (w^\varepsilon, w^\varepsilon_\tau)$, we get the rescaled travelling wave profile

$$w^\varepsilon\left(\tilde{\tau}, \xi\right) = y^\varepsilon(\varepsilon\tilde{\tau}) + \phi_1^\varepsilon\left(y^\varepsilon(\varepsilon\tilde{\tau}), \delta\right)(\xi),$$

and from the convergence results we have proved we understand what happens to these rescaled travelling wave solutions as $\varepsilon \to 0$.

5.5. **Convergence of Generalised Travelling Wave Solutions.** We now use this information to prove our desired convergence result for generalised travelling wave solutions.
Form the way we defined rescaled travelling wave solutions it follows that the travelling wave profile is of the form

$$v^\varepsilon\left(\tau, \xi\right) = w^\varepsilon\left(\frac{\tau}{\varepsilon}, \xi\right) = y^\varepsilon(\tau) + \phi_1^\varepsilon\left(y^\varepsilon(\tau), \delta\right)(\xi)$$

and therefore, from the convergence of the solutions y^ε and of the reduction map ϕ^ε it follows that

$$v^\varepsilon\left(\tau, \xi\right) = y^0(\tau) + O(\varepsilon) =: v^0(\tau) + O(\varepsilon) \text{ as } \varepsilon \to 0,$$

uniformly on $\mathbb{R}$, where v^0 is the limiting travelling wave profile and is a heteroclinic connection between equilibria for the ordinary differential equation,

$$v^0_\tau = -\frac{\delta v^0 + p_0 q(v^0)}{c}.$$

Thus we have proved the desired result of Theorem 2.3.

□

Acknowledgements. Financial support by EPSRC through DTA funding for Boden is gratefully acknowledged.

Appendix A. Calculation of Reduction Map for One Dimensional Centre Manifold

In order to calculate the equation on X_c up to quadratic order in U^c we need to calculate all the linear terms and the quadratic term on the U_0^- component of the reduction map ψ. Thus we need to satisfy

$$D_{(U_c,\delta)}\psi\left(U^c,\delta\right)\begin{pmatrix}\pi_c F\left(U^c+\psi\left(U^c,\delta\right),\delta\right)\\ 0\end{pmatrix}=\mathcal{A}\psi\left(U^c,\delta\right)+\pi_h F\left(U^c+\psi\left(U^c,\delta\right),\delta\right) \tag{53}$$

for these terms. We do this by expanding the reduction map in terms of the eigenfunctions and solving the equation on each eigenfunction component for the relevant terms, thus we write

$$\psi\left(U^c,\delta\right)=\psi_0^-\left(U^c,\delta\right)U_0^-+\sum_{m\in\mathbb{Z}^2\setminus\{0\}}\psi_m^\pm\left(U^c,\delta\right)U_m^\pm.$$

Now if we let $\psi_0^+\left(U^c,\delta\right):=u_1^c$ and suppress the arguments of the functions $\psi_m^\pm$, then we can obtain equations for the projections of F onto X_c and X_h

$$\pi_c F\left(U^c+\psi\left(U^c,\delta\right),\delta\right)=-\,\delta\left(\frac{\psi_0^++\psi_0^-}{\lambda_0^+-\lambda_0^-}\right)U_0^+ \tag{54}$$

$$-\left(\sum_{n,k\in\mathbb{Z}^2}q_2p_n\frac{\left(\psi_k^++\psi_k^-\right)\left(\psi_{-n-k}^++\psi_{-n-k}^-\right)}{\lambda_0^+-\lambda_0^-}\right)U_0^+ \tag{55}$$

and

$$\begin{aligned}\pi_h F\left(U^c+\psi\left(U^c,\delta\right),\delta\right)=&\delta\left(\frac{\psi_0^++\psi_0^-}{\lambda_0^+-\lambda_0^-}\right)U_0^-\\&+\left(\sum_{n,k\in\mathbb{Z}^2}q_2p_n\frac{\left(\psi_k^++\psi_k^-\right)\left(\psi_{-n-k}^++\psi_{-n-k}^-\right)}{\lambda_0^+-\lambda_0^-}\right)U_0^-\\&+\sum_{m\in\mathbb{Z}^2\setminus\{0\}}\mp\Bigg(\delta\left(\frac{\psi_m^++\psi_m^-}{\lambda_m^+-\lambda_m^-}\right)U_m^\pm+\\&\left(\sum_{n,k\in\mathbb{Z}^2}q_2p_n\frac{\left(\psi_k^++\psi_k^-\right)\left(\psi_{m-n-k}^++\psi_{m-n-k}^-\right)}{\lambda_m^+-\lambda_m^-}\right)U_m^\pm\Bigg);\end{aligned}$$

up to quadratic order in U^c by substituting this expansion into F.
We now want to calculate the linear terms of the Taylor expansion of ψ. Thus let

$$\psi\left(U^c,\delta\right)=\left(L_0^-(\delta)U^c\right)U_0^-+\sum_{m\in\mathbb{Z}^2\setminus\{0\}}\left(L_m^\pm(\delta)U^c\right)U_m^\pm+O\left(\|U^c\|^2\right),$$

where the functions $L_m^\pm(\delta):X_c\to\mathbb{C}$ are linear. Thus, if we substitute the equations for F on X_c and X_h into (53) and use the above expansion for ψ, we get that up to linear terms

$$\begin{aligned}&D_{(U^c,\delta)}\psi\left(U^c,\delta\right)\begin{pmatrix}\pi_c F\left(U^c+\psi\left(U^c,\delta\right),\delta\right)\\0\end{pmatrix}\\&\quad=L_0^-(\delta)\left(-\delta\left(\frac{u_1^c+L_0^-(\delta)U^c}{\lambda_0^+-\lambda_0^-}\right)U_0^+\right)U_0^-\\&\qquad+\sum_{m\in\mathbb{Z}\setminus\{0\}}L_m^\pm(\delta)\left(-\delta\left(\frac{u_1^c+L_0^-(\delta)U^c}{\lambda_0^+-\lambda_0^-}\right)U_0^+\right)U_m^\pm+O\left(\|U^c\|^2\right)\end{aligned}$$

and

$$
\begin{aligned}
&\mathcal{A}\psi\left(U^c,\delta\right)+\pi_h F\left(U^c+\psi\left(U^c,\delta\right),\delta\right)\\
&\qquad=\left(\lambda_0^- L_0^-(\delta)U^c\right)U_0^- + \sum_{m\in\mathbb{Z}^2\setminus\{0\}}\left(\lambda_m^\pm L_m^\pm(\delta)U^c\right)U_m^\pm\\
&\qquad\quad+\delta\left(\frac{u_1^c+L_0^-(\delta)U^c}{\lambda_0^+-\lambda_0^-}\right)U_0^- + \delta\sum_{m\in\mathbb{Z}^2\setminus\{0\}}\mp\left(\frac{L_m^+(\delta)U^c+L_m^-(\delta)U^c}{\lambda_m^+-\lambda_m^-}\right)U_m^\pm\\
&\qquad\quad+O\left(\|U^c\|^2\right).
\end{aligned}
$$

So on the U_0^- component of (53) we have the equation

$$
-\delta L_0^-(\delta)\left(\left(\frac{u_1^c+L_0^-(\delta)U^c}{\lambda_0^+-\lambda_0^-}\right)U_0^+\right)=\lambda_0^-\left(L_0^-(\delta)U^c\right)+\delta\left(\frac{u_1^c+L_0^-(\delta)U^c}{\lambda_0^+-\lambda_0^-}\right). \tag{56}
$$

Now, as $X_c=\mathrm{Span}_{\mathbb{R}}\left\{U_0^+\right\}$, we can write $U^c=yU_0^+$ for $y\in\mathbb{R}$ and $L_0^-(\delta)\left(yU_0^+\right):=a_0^-y$ for some $a_0^-\in\mathbb{R}$. Then, since $\lambda_0^+=0$ and $\lambda_0^-=-c$, the equation on the U_0^- component (56) becomes

$$
-\delta a_0^-\left(\frac{y+a_0^-y}{c}\right)=-ca_0^-y+\delta\left(\frac{y+a_0^-y}{c}\right),
$$

which can be rearranged to get

$$
\delta\left(1+a_0^-\right)^2=c^2a_0^-.
$$

Thus we get two possible values for a_0^-

$$
\begin{aligned}
a_0^- &= \frac{\left(c^2-2\delta\right)+\sqrt{\left(c^2-2\delta\right)^2-4\delta^2}}{2\delta}=\frac{c^2}{\delta}-2-\frac{2\delta}{c^2}+O(\delta^2)\text{ as }\delta\to 0\\
&\text{or}\\
a_0^- &= \frac{\left(c^2-2\delta\right)-\sqrt{\left(c^2-2\delta\right)^2-4\delta^2}}{2\delta}=\frac{2\delta}{c^2}+O(\delta^2)\text{ as }\delta\to 0;
\end{aligned}
$$

however if we choose the first of these then the reduction map ψ would not satisfy the condition

$$
D_{(U^c,\delta)}\psi(0,0)=0.
$$

So we take

$$
a_0^-:=\frac{\left(c^2-2\delta\right)-\sqrt{\left(c^2-2\delta\right)^2-4\delta^2}}{2\delta}.
$$

Now on the $U_m^\pm$ components of (53) for $m\neq 0$ we have the equation

$$
-\delta L_m^\pm(\delta)\left(\left(\frac{u_1^c+L_0^-(\delta)U^c}{\lambda_0^+-\lambda_0^-}\right)U_0^+\right)=\lambda_m^\pm\left(L_m^\pm(\delta)U^c\right)\mp\delta\left(\frac{L_m^+(\delta)U_c+L_m^-(\delta)U^c}{\lambda_m^+-\lambda_m^-}\right);
$$

which is satisfied if we take $L_m^\pm=0$. Hence we have determined the linear terms of the reduction map.
Next we want to find the quadratic terms on the U_0^- component. Thus we let

$$
\psi\left(U^c,\delta\right)=\left(L_0^-(\delta)U^c+Q_0^-\left(U^c,\delta\right)\right)U_0^-+\sum_{m\in\mathbb{Z}^2\setminus\{0\}}Q_m^\pm\left(U^c,\delta\right)+O\left(\|U^c\|^3\right);
$$

where $Q_m^\pm(\cdot,\delta):X_c\to\mathbb{C}$ are quadratic functions. Then, substituting the equations for F on X_c and X_h into (53) and using the above expansion for ψ in terms of the linear and quadratic terms, we get that the quadratic terms of

$$
\begin{aligned}
&D_{(U^c,\delta)}\psi\left(U^c,\delta\right)\begin{pmatrix} \pi_c F\left(U^c+\psi\left(U^c,\delta\right),\delta\right) \\ 0 \end{pmatrix} \\
&\quad = L_0^-\left(-\delta\left(\frac{Q_0^-\left(U^c,\delta\right)}{\lambda_0^+-\lambda_0^-}\right)U_0^+ - \frac{q_2p_0}{\lambda_0^+-\lambda_0^-}\left(u_1^c+L_0^-(\delta)U_c\right)^2U_0^+\right)U_0^- \\
&\quad + D_{U^c}Q_0^-\left(U^c,\delta\right)\left(-\delta\left(\frac{u_1^c+L_0^-(\delta)U_c}{\lambda_0^+-\lambda_0^-}\right)U_0^+\right)U_0^- \\
&\quad + \sum_{m\in\mathbb{Z}^2\setminus\{0\}} D_{U^c}Q_m^\pm\left(U^c,\delta\right)\left(-\delta\left(\frac{u_1^c+L_0^-(\delta)U_c}{\lambda_0^+-\lambda_0^-}\right)U_0^+\right)U_m^\pm
\end{aligned}
$$

and the quadratic terms of

$$
\begin{aligned}
&\mathcal{A}\psi\left(U^c,\delta\right)+\pi_h F\left(U^c+\psi\left(U^c,\delta\right),\delta\right) \\
&\quad = \lambda_0^-Q_0^-\left(U^c,\delta\right)U_0^- + \sum_{m\in\mathbb{Z}^2\setminus\{0\}}\lambda_m^\pm Q_m^\pm\left(U^c,\delta\right)U_m^\pm \\
&\quad + \delta\left(\frac{Q_0^-\left(U^c,\delta\right)}{\lambda_0^+-\lambda_0^-}\right)U_0^- + \frac{q_2p_0}{\lambda_0^+-\lambda_0^-}\left(u_1^c+L_0^-(\delta)U_c\right)^2U_0^- \\
&\quad + \sum_{m\in\mathbb{Z}^2\setminus\{0\}}\mp\Bigg(\delta\left(\frac{Q_m^+\left(U^c,\delta\right)+Q_m^-\left(U^c,\delta\right)}{\lambda_m^+-\lambda_m^-}\right)U_m^\pm + \\
&\quad \frac{q_2p_m}{\lambda_m^+-\lambda_m^-}\left(u_1^c+L_0^-(\delta)U^c\right)^2U_m^\pm\Bigg).
\end{aligned}
$$

Thus on the U_0^- component of (53) we have the equation

$$
\begin{aligned}
&L_0^-\left(-\delta\left(\frac{Q_0^-\left(U^c,\delta\right)}{\lambda_0^+-\lambda_0^-}\right)U_0^+ - \frac{q_2p_0}{\lambda_0^+-\lambda_0^-}\left(u_1^c+L_0^-(\delta)U_c\right)^2U_0^+\right) \\
&\quad + D_{U^c}Q_0^-\left(U^c,\delta\right)\left(-\delta\left(\frac{u_1^c+L_0^-(\delta)U_c}{\lambda_0^+-\lambda_0^-}\right)U_0^+\right) \\
&\quad = \lambda_0^-Q_0^-\left(U^c,\delta\right)+\delta\left(\frac{Q_0^-\left(U^c,\delta\right)}{\lambda_0^+-\lambda_0^-}\right)+\frac{q_2p_0}{\lambda_0^+-\lambda_0^-}\left(u_1^c+L_0^-(\delta)U_c\right)^2
\end{aligned}
\tag{57}
$$

and, since $X_c=\mathrm{Span}_{\mathbb{R}}\left\{U_0^+\right\}$, we can let $U^c=yU_0^+$, $L_0^-(\delta)\left(yU_0^+\right)=a_0^-y$ and $Q_0^-\left(yU_0^+,\delta\right)=b_0^-y^2$ for some $b_0^-\in\mathbb{R}$. Thus the equation on the U_0^- component (57) becomes

$$
\begin{aligned}
&-a_0^-\left(\frac{\delta b_0^-y^2+q_2p_0\left(y+a_0^-y\right)^2}{c}\right)-2b_0^-y\left(\delta\frac{\left(y+a_0^-y\right)}{c}\right) \\
&\quad = -cb_0^-y^2+\delta\frac{b_0^-y^2}{c}+\frac{q_2p_0}{c}\left(y+a_0^-y\right)^2;
\end{aligned}
$$

rearranging we get

$$
\left(c^2-3\delta\left(1+a_0^-\right)\right)b_0^-=q_2p_0\left(1+a_0^-\right)^3.
$$

Hence we have

$$
b_0^-=\frac{q_2p_0\left(1+a_0^-\right)^3}{c^2-3\delta\left(1+a_0^-\right)}.
$$

Thus we have determined the terms of the reduction map necessary to calculate the equation on the X_c up to quadratic order in U^c.

Appendix B. Calculation of Reduction Map for Two Dimensional Centre Manifold

In order to calculate the equation on X_c up to quadratic order in U^c we will need to calculate all the linear terms and the quadratic terms on the U_η^+ and $U_{-\eta}^+$ components of the reduction map. Thus we need to satisfy the equation

$$D_{(U^c,\delta)}\psi\left(U^c,\delta\right)\begin{pmatrix}\pi_c F\left(U^c+\psi\left(U^c,\delta\right),\delta\right)\\ 0\end{pmatrix}=\mathcal{A}\psi\left(U^c,\delta\right)+\pi_h F\left(U^c+\psi\left(U^c,\delta\right),\delta\right); \tag{53}$$

for these terms. We do this by expanding the reduction map in terms of the eigenfunctions and solving the equation on each eigenfunction component. Thus we let

$$\psi\left(U^c,\delta\right)=\psi_\eta^-\left(U^c,\delta\right)U_\eta^-+\psi_{-\eta}^-\left(U^c,\delta\right)U_{-\eta}^-+\sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\psi_m^\pm\left(U^c,\delta\right)U_m^\pm.$$

Now if we let $U_c=zU_\eta^++\overline{z}U_{-\eta}^+$ and, for notational convenience, set $\psi_\eta^+\left(U^c,\delta\right):=z$ and $\psi_{-\eta}^+\left(U^c,\delta\right):=\overline{z}$. Then, suppressing the arguments of the functions $\psi_m^\pm$ and substitute the above expansion into F, we get equations for the projections of F onto X_c and X_h,

$$\begin{aligned}\pi_c F\left(U^c+\psi\left(U^c,\delta\right),\delta\right)=&-\delta\left(\frac{\psi_\eta^++\psi_\eta^-}{\lambda_\eta^+-\lambda_\eta^-}\right)U_\eta^+-\delta\left(\frac{\psi_{-\eta}^++\psi_{-\eta}^-}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)U_{-\eta}^+\\&-\sum_{n,k\in\mathbb{Z}^2}q_2p_{\eta+n}\frac{\left(\psi_k^++\psi_k^-\right)\left(\psi_{-n-k}^++\psi_{-n-k}^-\right)}{\lambda_\eta^+-\lambda_\eta^-}U_\eta^+\\&-\sum_{n,k\in\mathbb{Z}^2}q_2p_{-\eta+n}\frac{\left(\psi_k^++\psi_k^-\right)\left(\psi_{-n-k}^++\psi_{-n-k}^-\right)}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}U_{-\eta}^+\end{aligned}\tag{58}$$

and

$$\begin{aligned}\pi_h F\left(U^c+\psi\left(U^c,\delta\right),\delta\right)=&\delta\left(\frac{\psi_\eta^++\psi_\eta^-}{\lambda_\eta^+-\lambda_\eta^-}\right)U_\eta^-+\delta\left(\frac{\psi_{-\eta}^++\psi_{-\eta}^-}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)U_{-\eta}^-\\&+\sum_{n,k\in\mathbb{Z}^2}q_2p_{\eta+n}\frac{\left(\psi_k^++\psi_k^-\right)\left(\psi_{-n-k}^++\psi_{-n-k}^-\right)}{\lambda_\eta^+-\lambda_\eta^-}U_\eta^-\\&+\sum_{n,k\in\mathbb{Z}^2}q_2p_{-\eta+n}\frac{\left(\psi_k^++\psi_k^-\right)\left(\psi_{-n-k}^++\psi_{-n-k}^-\right)}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}U_{-\eta}^-\\&+\sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\mp\Bigg(\delta\left(\frac{\psi_m^++\psi_m^-}{\lambda_m^+-\lambda_m^-}\right)U_m^\pm+\\&\sum_{n,k\in\mathbb{Z}^2}q_2p_{m+n}\frac{\left(\psi_k^++\psi_k^-\right)\left(\psi_{-k-n}^++\psi_{-k-n}^-\right)}{\lambda_m^+-\lambda_m^-}U_m^\pm\Bigg)\end{aligned}\tag{59}$$

up to quadratic order in U^c.
Now we start finding the terms of the reduction map ψ by working out the its linear terms, so we let

$$\psi\left(U^c,\delta\right)=\left(L_\eta^-(\delta)U^c\right)U_\eta^-+\left(L_{-\eta}^-(\delta)U^c\right)U_{-\eta}^-+\sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\left(L_m^\pm(\delta)U^c\right)U_m^\pm+O\left(\|U^c\|^2\right);$$

where the functions $L_m^\pm(\delta):X_c\to\mathbb{C}$ are linear. Now if we let $U^c=zU_\eta^++\overline{z}U_{-\eta}^+$ for $z\in\mathbb{C}$. Then using the equations for the projections of F onto X_c and X_h with the above expansion some calculation shows that up to linear terms in U^c

$$
\begin{aligned}
&D_{(U^c,\delta)}\psi\left(U^c,\delta\right)\begin{pmatrix}\pi_c F\left(U^c+\psi\left(U^c,\delta\right),\delta\right)\\ 0\end{pmatrix}\\
&\quad= L_\eta^-(\delta)\left(-\delta\left(\frac{z+L_\eta^-(\delta)U^c}{\lambda_\eta^+-\lambda_\eta^-}\right)U_\eta^+-\delta\left(\frac{\overline{z}+L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)U_{-\eta}^+\right)U_\eta^-\\
&\qquad+L_{-\eta}^-(\delta)\left(-\delta\left(\frac{z+L_\eta^-(\delta)U^c}{\lambda_\eta^+-\lambda_\eta^-}\right)U_\eta^+-\delta\left(\frac{\overline{z}+L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)U_{-\eta}^+\right)U_{-\eta}^-\\
&\qquad+\sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}L_m^\pm(\delta)\left(-\delta\left(\frac{z+L_\eta^-(\delta)U^c}{\lambda_\eta^+-\lambda_\eta^-}\right)U_\eta^+-\delta\left(\frac{\overline{z}+L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)U_{-\eta}^+\right)U_m^\pm
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{A}\psi\left(U^c,\delta\right)+\pi_h F\left(U^c+\psi\left(U^c,\delta\right),\delta\right)\\
&\qquad=\left(\lambda_\eta^-L_\eta^-(\delta)U^c\right)U_\eta^-+\left(\lambda_{-\eta}^-L_{-\eta}^-(\delta)U^c\right)U_{-\eta}^-+\sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\left(\lambda_m^\pm L_m^\pm(\delta)U^c\right)U_m^\pm\\
&\qquad\quad+\delta\left(\frac{z+L_\eta^-(\delta)U^c}{\lambda_\eta^+-\lambda_\eta^-}\right)U_\eta^-+\delta\left(\frac{\overline{z}+L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)U_{-\eta}^-\\
&\qquad\quad+\delta\sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\mp\left(\frac{L_m^+(\delta)U^c+L_m^-(\delta)U^c}{\lambda_m^+-\lambda_m^-}\right)U_m^\pm.
\end{aligned}
$$

Thus on the U_η^- and $U_{-\eta}^-$ components of (53) we have the equations

$$
\begin{aligned}
&L_\eta^-(\delta)\left(-\delta\left(\frac{z+L_\eta^-(\delta)U^c}{\lambda_\eta^+-\lambda_\eta^-}\right)U_\eta^+-\delta\left(\frac{\overline{z}+L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)U_{-\eta}^+\right)\\
&\qquad=\lambda_\eta^-L_\eta^-(\delta)U^c+\delta\left(\frac{z+L_\eta^-(\delta)U^c}{\lambda_\eta^+-\lambda_\eta^-}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&L_{-\eta}^-(\delta)\left(-\delta\left(\frac{z+L_\eta^-(\delta)U^c}{\lambda_\eta^+-\lambda_\eta^-}\right)U_\eta^+-\delta\left(\frac{\overline{z}+L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)U_{-\eta}^+\right)\\
&\qquad=\lambda_{-\eta}^-L_{-\eta}^-(\delta)U^c+\delta\left(\frac{z+L_{-\eta}^-(\delta)U^c}{\lambda_\eta^+-\lambda_\eta^-}\right).
\end{aligned}
$$

However, since ψ maps into $X_h\subset X$, we have that $L_\eta^-(\delta)=\overline{L_{-\eta}^-(\delta)}$ and it follows that the two equation above are complex conjugates of each other.
Now, since $U^c=zU_\eta^++\overline{z}U_{-\eta}^+$, we have that $L_\eta^-(\delta)\left(zU_\eta^++\overline{z}U_{-\eta}^+\right)=\alpha_\eta^-z+\beta_\eta^-\overline{z}$ for some $\alpha_\eta^-,\beta_\eta^-\in\mathbb{C}$ and we can express the equation on U_η^- in terms of z to get,

$$
\begin{aligned}
&-\delta\alpha_\eta^-\left(\frac{z+\left(\alpha_\eta^-z+\beta_\eta^-\overline{z}\right)}{\lambda_\eta^+-\lambda_\eta^-}\right)-\delta\beta_\eta^-\left(\frac{\overline{z}+\left(\overline{\beta_\eta^-}z+\overline{\alpha_\eta^-}\,\overline{z}\right)}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)\\
&\qquad=\lambda_\eta^-\left(\alpha_\eta^-z+\beta_\eta^-\overline{z}\right)+\delta\left(\frac{z+\left(\alpha_\eta^-z+\beta_\eta^-\overline{z}\right)}{\lambda_\eta^+-\lambda_\eta^-}\right).
\end{aligned}
$$

From this equation we can work out the coefficients α_η^- and β_η^- by equating the coefficients of the z and $\overline{z}$ terms,

$$z : -\delta\left(\frac{\alpha_\eta^-\left(1+\alpha_\eta^-\right)}{\lambda_\eta^+-\lambda_\eta^-}+\frac{\left|\beta_\eta^-\right|^2}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)=\lambda_\eta^-\alpha_\eta^-+\delta\left(\frac{1+\alpha_\eta^-}{\lambda_\eta^+-\lambda_\eta^-}\right)$$
$$\overline{z} : -\delta\left(\frac{\alpha_\eta^-\beta_\eta^-}{\lambda_\eta^+-\lambda_\eta^-}+\frac{\beta_\eta^-\left(1+\overline{\alpha_\eta^-}\right)}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}\right)=\lambda_\eta^-\beta_\eta^-+\delta\left(\frac{\beta_\eta^-}{\lambda_\eta^+-\lambda_\eta^-}\right).$$

Now the $\overline{z}$ equation implies that $\beta_\eta^-=0$, so rearranging the z equation we get the quadratic equation

$$\delta\left(\alpha_\eta^-\right)^2+\left(2\delta+\lambda_\eta^-\left(\lambda_\eta^+-\lambda_\eta^-\right)\right)\alpha_\eta^-+\delta=0.$$

So we have two possible values for α_η^- namely,

$$\begin{aligned}\alpha_\eta^- &= \frac{-\left(2\delta+\lambda_\eta^-\left(\lambda_\eta^+-\lambda_\eta^-\right)\right)+\sqrt{\left(2\delta+\lambda_\eta^-\left(\lambda_\eta^+-\lambda_\eta^-\right)\right)^2-4\delta^2}}{2\delta}\\ &= O(\delta) \text{ as } \delta\to 0\end{aligned}$$

or

$$\begin{aligned}\alpha_\eta^- &= \frac{-\left(2\delta+\lambda_\eta^-\left(\lambda_\eta^+-\lambda_\eta^-\right)\right)-\sqrt{\left(2\delta+\lambda_\eta^-\left(\lambda_\eta^+-\lambda_\eta^-\right)\right)^2-4\delta^2}}{2\delta}\\ &= -\frac{\lambda_\eta^-\left(\lambda_\eta^+-\lambda_\eta^-\right)}{2\delta}-2+O(\delta) \text{ as } \delta\to 0.\end{aligned}$$

However if we choose the second of these values then the reduction map would not satisfy the condition,

$$D_{(U^c,\delta)}\psi(0,0)=0.$$

Thus we choose the first of the values given above and hence we have determined $L_\eta^-(\delta)$ and $L_{-\eta}^-(\delta)$, since $L_{-\eta}^-(\delta)$ is the complex conjugate of $L_\eta^-(\delta)$.
Now on the $U_m^\pm$ component of equation (53) for $m\neq\pm\eta$ we have the equation,

$$\begin{aligned}-\delta L_m^\pm(\delta)&\left(\frac{z+L_\eta^-(\delta)U_c}{\lambda_\eta^+-\lambda_\eta^-}U_\eta^++\frac{\overline{z}+L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}U_{-\eta}^+\right)\\ &=\lambda_m^\pm L_m^\pm(\delta)U^c\mp\delta\left(\frac{L_m^+(\delta)U^c+L_m^-(\delta)U^c}{\lambda_m^+-\lambda_m^-}\right);\end{aligned}$$

which is satisfied if we choose $L_m^\pm(\delta)=0$ for all $m\neq\pm\eta$. Thus we have determined all the linear terms of the reduction map, we summaries this information below;

$$\begin{aligned}L_\eta^-(\delta)\left(zU_\eta^+\overline{z}U_\eta^+\right)&=\alpha_\eta^-z\\ L_{-\eta}^-(\delta)\left(zU_\eta^+\overline{z}U_\eta^+\right)&=\overline{\alpha_\eta^-}\,\overline{z}\\ L_m^\pm(\delta)\left(zU_\eta^+\overline{z}U_\eta^+\right)&=0 \text{ for all } m\neq\pm\eta.\end{aligned}$$

Hence it just remains to determine the quadratic terms on the U_η^- and $U_{-\eta}^-$ components. Thus we let

$$\begin{aligned}\psi\left(U^c,\delta\right)=&\left(L_\eta^-(\delta)U^c+Q_\eta^-\left(U^c,\delta\right)\right)U_{-\eta}^-+\left(L_{-\eta}^-(\delta)U^c+Q_{-\eta}^-\left(U^c,\delta\right)\right)U_{-\eta}^-\\ &+\sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}Q_m^\pm\left(U^c,\delta\right)U_m^\pm+O\left(\|U^c\|^3\right);\end{aligned}$$

where $Q_m^\pm(\cdot,\delta):X_c\to\delta$ are quadratic functions. Then, if we let $U^c=zU_\eta^-+\overline{z}U_{-\eta}^+$ and use the equation for the projections of F onto X_c and X_h with the above expansion, we get that the quadratic terms in U^c of

$$
\begin{aligned}
&D_{(U^c,\delta)}\psi\left(U^c,\delta\right)\begin{pmatrix} \pi_c F\left(U^c+\psi\left(U^c,\delta\right),\delta\right) \\ 0 \end{pmatrix} \\
&\qquad = \left(L_\eta^-(\delta)V\right)U_\eta^- + \left(L_{-\eta}^-(\delta)V\right)U_{-\eta}^- + \left(D_{U^c}Q_\eta^-\left(U^c,\delta\right)W\right)U_\eta^- \\
&\qquad\quad + \left(D_{U^c}Q_{-\eta}^-\left(U^c,\delta\right)W\right)U_{-\eta}^- + \sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\left(D_{U^c}Q_m^\pm\left(U^c,\delta\right)W\right)U_m^\pm;
\end{aligned}
$$

where

$$
\begin{aligned}
V = &-\frac{\delta Q_\eta^-\left(U^c,\delta\right)}{\lambda_\eta^+-\lambda_\eta^-}U_\eta^+ - \frac{\delta Q_{-\eta}^-\left(U^c,\delta\right)}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}U_{-\eta}^+ \\
&- \frac{q_2\left(p_{-\eta}\left(z+L_\eta^-(\delta)U^c\right)^2 + 2p_\eta\left|z+L_\eta^-(\delta)U^c\right|^2 + p_{3\eta}\left(\overline{z}+L_{-\eta}^-(\delta)U^c\right)^2\right)}{\lambda_\eta^+-\lambda_\eta^-}U_\eta^+ \\
&- \frac{q_2\left(p_{-3\eta}\left(z+L_\eta^-(\delta)U^c\right)^2 + 2p_{-\eta}\left|z+L_\eta^-(\delta)U^c\right|^2 + p_\eta\left(\overline{z}+L_{-\eta}^-(\delta)U^c\right)^2\right)}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}U_{-\eta}^+
\end{aligned}
$$

and

$$
W = -\delta\left(\frac{z+L_\eta^-U^c}{\lambda_\eta^+-\lambda_\eta^-}\right)U_\eta^+ - \delta\left(\frac{\overline{z}+L_{-\eta}^-U^c}{\lambda_{-\eta}^+-\lambda_{-\eta}}\right)U_{-\eta}^+,
$$

and on the other hand using the same method we get that the quadratic terms in U^c of

$$
\begin{aligned}
&\mathcal{A}\psi\left(U^c,\delta\right) + \pi_h F\left(U^c+\psi\left(U^c,\delta\right),\delta\right) \\
&= \left(\lambda_\eta^- Q_\eta^-\left(U^c,\delta\right)\right)U_\eta^- + \left(\lambda_{-\eta}^- Q_{-\eta}^-\left(U^c,\delta\right)\right)U_{-\eta}^- + \sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\left(\lambda_m^\pm Q_m^p m\left(U^c,\delta\right)\right)U_m^\pm \\
&+ \tilde{V} + \sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\mp\Bigg(\delta\left(\frac{Q_m^+\left(U^c,\delta\right)+Q_m^-\left(U^c,\delta\right)}{\lambda_m^+-\lambda_m^-}\right)U_m^\pm \\
&+ q_2\left(\frac{p_{m-2\eta}\left(z+L_\eta^-(\delta)U^c\right)^2 + p_m\left|z+L_\eta^-(\delta)U^c\right|^2 + p_{m+2\eta}\left(\overline{z}+L_{-\eta}^-(\delta)U^c\right)^2}{\lambda_m^+-\lambda_m^-}\right)U_m^\pm\Bigg);
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{V} = &\frac{\delta Q_\eta^-\left(U^c,\delta\right)}{\lambda_\eta^+-\lambda_\eta^-}U_\eta^- + \frac{\delta Q_{-\eta}^-\left(U^c,\delta\right)}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}U_{-\eta}^- \\
&+ \frac{q_2\left(p_{-\eta}\left(z+L_\eta^-(\delta)U^c\right)^2 + 2p_\eta\left|z+L_\eta^-(\delta)U^c\right|^2 + p_{3\eta}\left(\overline{z}+L_{-\eta}^-(\delta)U^c\right)^2\right)}{\lambda_\eta^+-\lambda_\eta^-}U_\eta^- \\
&+ \frac{q_2\left(p_{-3\eta}\left(z+L_\eta^-(\delta)U^c\right)^2 + 2p_{-\eta}\left|z+L_\eta^-(\delta)U^c\right|^2 + p_\eta\left(\overline{z}+L_{-\eta}^-(\delta)U^c\right)^2\right)}{\lambda_{-\eta}^+-\lambda_{-\eta}^-}U_{-\eta}^-.
\end{aligned}
$$

Now, as we only want to workout Q_η^- and $Q_{-\eta}^-$ we can just look at the equation on the U_η^- component, as $\overline{Q_\eta^-} = Q_{-\eta}^-$. The equation on the U_η^- component is

$$
\begin{aligned}
&L_\eta^-(\delta)V + D_{U^c}Q_\eta^-\left(U^c,\delta\right)W \\
&\qquad = \lambda_\eta^- Q_\eta^-\left(U^c,\delta\right) + \frac{\delta Q_\eta^-\left(U^c,\delta\right)}{\lambda_\eta^+-\lambda_\eta^-} \\
&\qquad\quad + \frac{q_2\left(p_{-\eta}\left(z+L_\eta^-(\delta)U^c\right)^2 + 2p_\eta\left|z+L_\eta^-(\delta)U^c\right|^2 + p_{3\eta}\left(\overline{z}+L_{-\eta}^-(\delta)U^c\right)^2\right)}{\lambda_\eta^+-\lambda_\eta^-}.
\end{aligned}
$$

Now we know that $L_\eta^-(\delta)\left(zU_\eta^+ + \overline{z}U_{-\eta}^+\right) = \alpha_\eta^- z$ and we can write $Q_\eta^-\left(zU_\eta^+ + \overline{z}U_{-\eta}^+\right) = \gamma_\eta^- z^2 + 2\zeta_\eta^- |z|^2 + \sigma_\eta^- \overline{z}^2$ for some $\gamma_\eta^-, \zeta_\eta^-, \sigma_\eta^- \in \mathbb{C}$. So therefore the above equation can be written in terms of z as

$$
\begin{aligned}
&-\frac{\alpha_\eta^-}{\lambda_\eta^+ - \lambda_\eta^-}\Bigg(\delta\left(\gamma_\eta^- z^2 + 2\zeta_\eta^- |z|^2 + \sigma_\eta^- \overline{z}^2\right) + q_2\Bigg(p_{-\eta}\left(1+\alpha_\eta^-\right)^2 z^2 + 2p_\eta\left|1+\alpha_\eta^-\right|^2|z|^2 \\
&+ p_{3\eta}\left(1+\overline{\alpha_\eta^-}\right)^2\overline{z}^2\Bigg)\Bigg) - 2\delta\left(\frac{\left(\gamma_\eta^- z + \zeta_\eta^- \overline{z}\right)\left(1+\alpha_\eta^-\right) z}{\lambda_\eta^+ - \lambda_\eta^-} + \frac{\left(\zeta_\eta^- z + \sigma_\eta^- \overline{z}\right)\left(1+\overline{\alpha_\eta^-}\right)\overline{z}}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right) \\
&= \lambda_\eta^-\left(\gamma_\eta^- z^2 + 2\zeta_\eta^- |z|^2 + \sigma_\eta^- \overline{z}^2\right) + \frac{1}{\lambda_\eta^+ - \lambda_\eta^-}\Bigg(\delta\left(\gamma_\eta^- z^2 + 2\zeta_\eta^- |z|^2 + \sigma_\eta^- \overline{z}^2\right) \\
&\quad + q_2\left(p_{-\eta}\left(1+\alpha_\eta^-\right)^2 z^2 + 2p_\eta\left|1+\alpha_\eta^-\right|^2|z|^2 + p_{3\eta}\left(1+\overline{\alpha_\eta^-}\right)^2\overline{z}^2\right)\Bigg),
\end{aligned}
$$

and if we now equate the coefficients of the z^2, $|z|^2$ and $\overline{z}^2$ terms and rearrange we get the following three equations.

$$
\begin{aligned}
z^2 &: \left(\lambda_\eta^-\left(\lambda_\eta^+ - \lambda_\eta^-\right) + 3\delta\left(1+\alpha_\eta^-\right)\right)\gamma_\eta^- = -q_2 p_{-\eta}\left(1+\alpha_\eta^-\right)^3 \\
|z|^2 &: \left(\lambda_\eta^-\left(\lambda_\eta^+ - \lambda_\eta^-\right) + 2\delta\left(1+\alpha_\eta^-\right) + \delta\frac{\left(1+\overline{\alpha_\eta^-}\right)\left(\lambda_\eta^+ - \lambda_\eta^-\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right)\zeta_\eta^- \\
&\qquad = -q_2 p_\eta\left(1+\alpha_\eta^-\right)^2\left(1+\overline{\alpha_\eta^-}\right) \\
\overline{z}^2 &: \left(\lambda_\eta^-\left(\lambda_\eta^+ - \lambda_\eta^-\right) + \delta\left(1+\alpha_\eta^-\right) + 2\delta\frac{\left(1+\overline{\alpha_\eta^-}\right)\left(\lambda_\eta^+ - \lambda_\eta^-\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right)\sigma_\eta^- \\
&\qquad = -q_2 p_{3\eta}\left(1+\alpha_\eta^-\right)\left(1+\overline{\alpha_\eta^-}\right)^2
\end{aligned}
$$

These three equations determine γ_η^-, ζ_η^- and σ_η^- and thus we have determined $Q_\eta^-\left(U^c, \delta\right)$ and $Q_{-\eta}^-\left(U^c, \delta\right)$ to be

$$
\begin{aligned}
Q_\eta^-\left(zU_\eta^+ + \overline{z}U_{-\eta}^+, \delta\right) &= \gamma_\eta^- z^2 + 2\zeta_\eta^- |z|^2 + \sigma_\eta^- \overline{z}^2 \\
Q_{-\eta}^-\left(zU_\eta^+ + \overline{z}U_{-\eta}^+, \delta\right) &= \overline{\sigma_\eta^-} z^2 + 2\overline{\zeta_\eta^-} |z|^2 + \overline{\gamma_\eta^-}\,\overline{z}^2.
\end{aligned}
$$

Thus we have determined the terms of the reduction map needed to calculate the ordinary differential equation on $X_c \times \mathbb{R}$ up to quadratic order in U^c.

References

[1] A. Afendikov and A. Mielke, *Bifurcations of Poiseuille flow between parallel plates: three-dimensional solutions with large spanwise wavelength*, Arch. Rational Mech. Anal., 129 (1995), pp. 101–127.

[2] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, AMS Chelsea Publishing, Providence, RI, 2011. Corrected reprint of the 1978 original.

[3] H. Berestycki and F. Hamel, *Front propagation in periodic excitable media*, Comm. Pure Appl. Math., 55 (2002), pp. 949–1032.

[4] H. Berestycki and L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat. (N.S.), 22 (1991), pp. 1–37.

[5] A. Boden, *Travelling Waves in Heterogeneous Media*, PhD thesis, University of Bath, 2012.

[6] D. Cioranescu and P. Donato, *An introduction to homogenization*, vol. 17 of Oxford Lecture Series in Mathematics and its Applications, The Clarendon Press Oxford University Press, New York, 1999.

[7] C. Conley, *Isolated invariant sets and the Morse index*, vol. 38 of CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, R.I., 1978.

[8] J.-P. Eckmann and C. E. Wayne, *Propagating fronts and the center manifold theorem*, Comm. Math. Phys., 136 (1991), pp. 285–307.

[9] B. Fiedler and A. Scheel, *Spatio-temporal dynamics of reaction-diffusion patterns*, in Trends in nonlinear analysis, Springer, Berlin, 2003, pp. 23–152.

[10] R. A. Fisher, *The advance of advantageous genes*, Ann. Eugenics., 7 (1937), pp. 335–369.
[11] M. Haragus and G. Iooss, *Local bifurcations, center manifolds, and normal forms in infinite-dimensional dynamical systems*, Universitext, Springer-Verlag London Ltd., London, 2011.
[12] M. Haragus and G. Schneider, *Bifurcating fronts for the Taylor-Couette problem in infinite cylinders*, Z. Angew. Math. Phys., 50 (1999), pp. 120–151.
[13] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian.
[14] K. Kirchgässner, *Wave-solutions of reversible systems and applications*, J. Differential Equations, 45 (1982), pp. 113–127.
[15] A. Kolmogorov, I. Petrovsky, and N. Piskunov, *Study of the diffusion equation with growth of the quantity of matter and its application to a biological problem*, Bull. Univ. Etat. Moscow Ser. Internat. Math. Mec. Sect. A., 1 (1937), pp. 1–29.
[16] K. Matthies, G. Schneider, and H. Uecker, *Exponential averaging for traveling wave solutions in rapidly varying periodic media*, Math. Nachr., 280 (2007), pp. 408–422.
[17] K. Matthies and C. E. Wayne, *Wave pinning in strips*, Proc. Roy. Soc. Edinburgh Sect. A, 136 (2006), pp. 971–995.
[18] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
[19] J. A. Sanders, F. Verhulst, and J. Murdock, *Averaging methods in nonlinear dynamical systems*, vol. 59 of Applied Mathematical Sciences, Springer, New York, second ed., 2007.
[20] B. Sandstede and A. Scheel, *Defects in oscillatory media: toward a classification*, SIAM J. Appl. Dyn. Syst., 3 (2004), pp. 1–68 (electronic).
[21] G. Schneider and H. Uecker, *Existence and stability of modulating pulse solutions in Maxwell's equations describing nonlinear optics*, Z. Angew. Math. Phys., 54 (2003), pp. 677–712.
[22] J. Smoller, *Shock waves and reaction-diffusion equations*, vol. 258 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], Springer-Verlag, New York, 1983.
[23] M. E. Taylor, *Partial differential equations. III*, vol. 117 of Applied Mathematical Sciences, Springer-Verlag, New York, 1997. Nonlinear equations.
[24] A. Vanderbauwhede and G. Iooss, *Center manifold theory in infinite dimensions*, 1 (1992), pp. 125–163.
[25] J. Xin, *Front propagation in heterogeneous media*, SIAM Rev., 42 (2000), pp. 161–230 (electronic).
[26] J. Xin, *An introduction to fronts in random media*, vol. 5 of Surveys and Tutorials in the Applied Mathematical Sciences, Springer, New York, 2009.

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom
E-mail address: adamcboden@gmail.com

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom
E-mail address: k.matthies@bath.ac.uk